

Dirichlet heat kernel estimates for fractional Laplacian under non-local perturbation

Zhen-Qing Chen^{*} and Ting Yang[†]

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Abstract

For $d \geq 2$ and $0 < \beta < \alpha < 2$, consider a family of non-local operators $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$ on \mathbb{R}^d , where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{A}(d, -\beta) \int_{\{z \in \mathbb{R}^d: |z| > \varepsilon\}} (f(x+z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} dz,$$

and $b(x, z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x, z) = b(x, -z)$ for every $x, z \in \mathbb{R}^d$. Here $\mathcal{A}(d, -\beta)$ is a normalizing constant so that $\mathcal{S}^b = -(-\Delta)^{\beta/2}$ when $b(x, z) \equiv 1$. It was recently shown in Chen and Wang [12] that when $b(x, z) \geq -\frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\alpha}$, \mathcal{L}^b admits a unique fundamental solution $p^b(t, x, y)$ which is strictly positive and continuous. The kernel $p^b(t, x, y)$ uniquely determines a conservative Feller process X^b , which has strong Feller property. The Feller process X^b is also the unique solution to the martingale problem of $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on \mathbb{R}^d . In this paper, we are concerned with the subprocess $X^{b,D}$ of X^b killed upon leaving a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$. We establish explicit sharp two-sided estimates for the transition density function of $X^{b,D}$.

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1 Introduction

Discontinuous Markov processes and non-local operators have been under intense study recently, due to their importance both in theory and in applications. Many physical and economic

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systems have been successfully modeled by non-Gaussian jump processes. The infinitesimal generator of a discontinuous Markov process in \mathbb{R}^d is no longer a differential operator but rather a non-local (or, integro-differential) operator. For instance, the infinitesimal generator of an isotropically symmetric α -stable process in \mathbb{R}^d with $\alpha \in (0, 2)$ is a fractional Laplacian operator $c\Delta^{\alpha/2} := -c(-\Delta)^{\alpha/2}$. During the past several years there is also much interest from the theory of PDE (such as singular obstacle problems) to study non-local operators; see, for example, [4] and the references therein.

Transition density function, also called heat kernel, of a Markov process encodes all the information about the process. However unless in some very special cases, the explicit formula of the transition density function is very difficult, if not impossible, to derive. Unlike the case for diffusion processes, two-sided heat kernel estimates for jump-diffusions in \mathbb{R}^d have only been systematically studied since around 2000. The study of the transition density function (also called Dirichlet heat kernel) of the subprocesses of jump-diffusions in open sets is even more recently. We refer the reader to [5] for a recent survey on this subject.

For heat kernel estimates of discontinuous Markov processes, most of work is restricted to symmetric Markov processes. In a recent paper [12], Chen and Wang studied the following class of non-symmetric non-local operators, that is, fractional Laplacian under non-local perturbations. Let $d \geq 2$ and $0 < \beta < \alpha < 2$. Consider non-local operator $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{A}(d, -\beta) \int_{\{z \in \mathbb{R}^d: |z| > \varepsilon\}} (f(x+z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} dz, \quad (1.1)$$

where $\mathcal{A}(d, -\beta) = \beta 2^{\beta-1} \pi^{-d/2} \Gamma((d+\beta)/2) \Gamma(1-\beta/2)^{-1}$ is the normalizing constant so that $\mathcal{S}^b = \Delta^{\beta/2} := -(-\Delta)^{\beta/2}$ when $b(x, z) \equiv 1$, and $b(x, z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$b(x, z) = b(x, -z) \quad \text{for } x, z \in \mathbb{R}^d, \quad (1.2)$$

In other words,

$$\mathcal{L}^b f(x) = \int_{\mathbb{R}^d} (f(y) - f(x) - \langle \nabla f(x), y - x \rangle \mathbb{1}_{\{|y-x| \leq 1\}}) j^b(x, y) dy,$$

where

$$j^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|y-x|^{d+\alpha}} \left(1 + \frac{\mathcal{A}(d, -\beta)}{\mathcal{A}(d, -\alpha)} b(x, y-x) |y-x|^{\alpha-\beta} \right). \quad (1.3)$$

It is established in [12] that if

$$\text{for every } x \in \mathbb{R}^d, \quad j^b(x, y) \geq 0 \quad \text{for a.e. } y \in \mathbb{R}^d \quad (1.4)$$

(that is, if for every $x \in \mathbb{R}^d$, $b(x, z) \geq -\frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\alpha}$ a.e. $z \in \mathbb{R}^d$), then \mathcal{L}^b admits a unique fundamental solution $p^b(t, x, y)$, which is strictly positive and jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. The kernel $p^b(t, x, y)$ uniquely determines a conservative strong Feller process X^b on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ such that

$$\mathbb{E}_x \left[f(X_t^b) \right] = \int_{\mathbb{R}^d} f(y) p^b(t, x, y) dy$$

for every bounded measurable function f on \mathbb{R}^d . Various explicit form of sharp two-sided estimates on $p^b(t, x, y)$ are obtained in [12]; see Proposition 2.1 for a partial summary. In this paper, we study the Dirichlet heat kernel estimates for \mathcal{L}^b in bounded $C^{1,1}$ open sets and their sharp two-sided estimates. As a consequence, we obtain sharp two-sided estimates on the Green function of \mathcal{L}^b in bounded $C^{1,1}$ open sets. To present the main results of this paper, we need first to recall some facts and notations.

In this paper we use “ $:=$ ” as a way of definition. We define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For any two positive functions f and g , $f \stackrel{c}{\lesssim} g$ means that there is a positive constant c such that $f \leq cg$ on their common domain of definition, and $f \stackrel{c}{\gtrsim} g$ means that $c^{-1}g \leq f \leq cg$. We also write “ \lesssim ” and “ \gtrsim ” if c is unimportant or understood. We use $B(x, r)$ to denote the open ball centered at x with radius $r > 0$. Let $\delta_D(x)$ denote the Euclidean distance between x and ∂D . We will use capital letters C_0, C_1, C_2, \dots to denote constants in the statements of results. The lower case constants c_0, c_1, c_2, \dots can change from one appearance to another. We will use dx to denote the Lebesgue measure in \mathbb{R}^d and $\text{diam}(D)$ to denote the diameter of D .

The Feller processes X^b correspond to \mathcal{L}^b contain non-local perturbations of several important Lévy processes. Observe that when $b \equiv 0$, then X^b is the (rotationally) symmetric α -stable process on \mathbb{R}^d . We denote its transition density function by $p(t, x, y)$. When $b \equiv a$ for some constant $a > 0$, then $\mathcal{L}^b = \Delta^{\alpha/2} + a\Delta^{\beta/2}$ and X^b is the independent sum of a symmetric α -stable process and a scaled symmetric β -stable process. Denote by $p_a(t, x, y)$ the corresponding transition density. It is proved in [7] that

$$p_a(t, x, y) \asymp \left(t^{-\frac{d}{\alpha}} \wedge (at)^{-\frac{d}{\beta}} \right) \wedge \left(\frac{t}{|x - y|^{d+\alpha}} + \frac{at}{|x - y|^{d+\beta}} \right)$$

on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. When $b(x, z) = -\frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)}|z|^{\beta-\alpha}1_{\{|z| \geq 1\}}$, X^b is a truncated symmetric α -stable process with Lévy intensity $\mathcal{A}(d, -\alpha)|x|^{-d-\alpha}1_{\{|x| < 1\}}dx$. Denote by $\bar{p}_1(t, x, y)$ its transition density function. It is proved in [8] that for $t \in (0, 1]$ and $|x - y| \leq 1$,

$$\bar{p}_1(t, x, y) \asymp t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}},$$

while for $t \in (0, 1]$ and $|x - y| > 1$.

$$c_1 \left(\frac{t}{|x - y|} \right)^{c_2|x-y|} \leq \bar{p}_1(t, x, y) \leq c_3 \left(\frac{t}{|x - y|} \right)^{c_4|x-y|}$$

for some constants $c_i = c_i(d, \alpha) > 0$, $i = 1, \dots, 4$.

For an open set $D \subset \mathbb{R}^d$, define $\tau_D^b := \inf\{t > 0 : X_t^b \notin D\}$. We will use $X^{b,D}$ to denote the subprocess of X^b killed upon leaving D , that is, $X_t^{b,D}(\omega) = X_t^b(\omega)$ if $t < \tau_D^b(\omega)$ and $X_t^{b,D}(\omega) = \partial$ if $t \geq \tau_D^b(\omega)$, where ∂ is a cemetery state. We use the convention that for every function f , we extend its definition to ∂ by setting $f(\partial) = 0$. Define

$$p_D^b(t, x, y) := p^b(t, x, y) - \mathbb{E}_x \left[p^b(t - \tau_D^b, X_{\tau_D^b}^b, y); \tau_D^b < t \right]. \quad (1.5)$$

Then $p_D^b(t, x, y)$ is the transition density of the subprocess $X^{b,D}$. It follows easily from the estimate of $p^b(t, x, y)$ (see Proposition 2.1 and Theorem 3.2 below) that the transition semigroup $\{P_t^{b,D}; t \geq 0\}$ of $X^{b,D}$, defined by $P_t^{b,D}f(x) = \mathbb{E}_x[f(X_t^{b,D})]$, is a strongly continuous semigroup in $L^2(D; dx)$. We use $\mathcal{L}^{b,D}$ to denote the infinitesimal generator of $\{P_t^{b,D}; t \geq 0\}$ in $L^2(D; dx)$. Intuitively, $\mathcal{L}^{b,D}$ is the operator \mathcal{L}^b in D with zero Dirichlet exterior condition on D^c . The (complex) spectrum of $\mathcal{L}^{b,D}$ is denoted by $\sigma(\mathcal{L}^{b,D})$; see Section 7 for its definition. For a complex number z , $\operatorname{Re} z$ denotes its real part.

Definition 1.1. An open set D in \mathbb{R}^d is said to be $C^{1,1}$ if there exists a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$, such that for any $Q \in \partial D$, there exists a $C^{1,1}$ function $\phi = \phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla \phi(0) = 0$, $\|\nabla \phi\|_\infty \leq \Lambda_0$, $|\nabla \phi(x) - \nabla \phi(y)| \leq \Lambda_0|x - y|$, and an orthonormal coordinate system CS_Q with its origin at Q such that

$$B(Q, R_0) \cap D = \{y = (\tilde{y}, y_d) \text{ in } CS_Q : |\tilde{y}| < R_0, y_d > \phi(\tilde{y})\}.$$

The pair (R_0, Λ_0) is called the $C^{1,1}$ characteristic of D .

The following is the main result of this paper.

Theorem 1.2. Let D be a bounded $C^{1,1}$ open subset of \mathbb{R}^d . Define

$$f_D(t, x, y) = \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

The following holds.

- (i) For every $A, T \in (0, \infty)$, there are positive constants $\lambda_0 = \lambda_0(d, \alpha, \beta, D, A)$ and $C_0 = C_0(d, \alpha, \beta, D, A, T)$ so that for any bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$,

$$p_D^b(t, x, y) \leq C_0 f_D(t, x, y) \quad \text{on } (0, T] \times D \times D$$

and

$$p_D^b(t, x, y) \leq C_0 e^{-t\lambda_0} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D.$$

Moreover, for every $b(x, z)$ satisfying the above conditions, $\lambda_1^{b,D} := -\sup \operatorname{Re} \sigma(\mathcal{L}^{b,D}) \geq \lambda_0$ and there is a positive constant $C_1 = C_1(d, \alpha, \beta, D, A, b, T)$ such that

$$p_D^b(t, x, y) \leq C_1 e^{-t\lambda_1^{b,D}} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D.$$

- (ii) For every $A, T \in (0, \infty)$, there are positive constants $r_1 = r_1(d, \alpha, \beta, A)$ and $C_i = C_i(d, \alpha, \beta, D, A, T)$, $i = 2, 3$, such that for any bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$, and any $x, y \in D$ with $|x - y| < r_1$,

$$p_D^b(t, x, y) \geq C_2 f_D(t, x, y) \quad \text{for } t \in (0, T],$$

$$p_D^b(t, x, y) \geq C_2 e^{-t\lambda_1^{b, D_x \cup D_y}} \delta_D(x) \delta_D(y) \quad \text{for } t \in (T, \infty),$$

where D_x denotes the connected component of D that contains x , and

$$\lambda_1^{b, D_x \cup D_y} := -\sup \operatorname{Re} \sigma(\mathcal{L}^{b, D_x \cup D_y}) > 0.$$

Suppose, in addition, D is connected, or the distance between any two connected components of D is less than r_1 , or the diameter of D is less than r_1 . Then

$$p_D^b(t, x, y) \geq C_3 f_D(t, x, y) \quad \text{on } (0, T] \times D \times D,$$

$$p_D^b(t, x, y) \geq C_3 e^{-t\lambda_1^{b, D}} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D.$$

(iii) For every $A, T, \varepsilon \in (0, \infty)$, there are positive constants $C_i = C_i(d, \alpha, \beta, D, A, T, \varepsilon) \geq 1$, $i = 4, 5$, such that for any bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and

$$j^b(x, y) \geq \varepsilon |y - x|^{-d-\alpha} \quad \text{for a.e. } x, y \in \mathbb{R}^d \quad (1.6)$$

with $\|b\|_\infty \leq A$, we have

$$C_4^{-1} f_D(t, x, y) \leq p_D^b(t, x, y) \leq C_4 f_D(t, x, y) \quad \text{on } (0, T] \times D \times D,$$

$$C_5^{-1} e^{-t\lambda_1^{b, D}} \delta_D(x) \delta_D(y) \leq p_D^b(t, x, y) \leq C_5 e^{-t\lambda_1^{b, D}} \delta_D(x) \delta_D(y) \quad \text{on } (T, \infty) \times D \times D,$$

where $\lambda_1^{b, D} := -\sup \operatorname{Re} \sigma(\mathcal{L}^{b, D}) > 0$.

Integrating in t of the above heat kernel estimates, we get the following sharp two-sided estimate on the Green function $G_D^b(x, y)$ of \mathcal{L}^b , since $G_D^b(x, y) = \int_0^\infty p_D^b(t, x, y) dt$. See the proof of [9, Corollary 1.2] for the details about such integration.

Corollary 1.3. For every $A, \varepsilon \in (0, \infty)$, there exists a constant $C_6 = C_6(d, \alpha, \beta, D, A, \varepsilon) \geq 1$, so that for any bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.6) with $\|b\|_\infty \leq A$,

$$\frac{C_6^{-1}}{|x - y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x) \delta_D(y)}{|x - y|^2} \right)^{\alpha/2} \leq G_D^b(x, y) \leq \frac{C_6}{|x - y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x) \delta_D(y)}{|x - y|^2} \right)^{\alpha/2}$$

for $(x, y) \in D \times D$.

We now describe the approach of this paper. Since $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$, heuristically $p_D^b(t, x, y)$ should relate to $p_D(t, x, y)$, the heat kernel of the killed symmetric α -stable process $X^{0, D}$ in D by

$$p_D^b(t, x, y) = p_D(t, x, y) + \int_0^t \int_D p_D^b(s, x, z) \mathcal{S}_z^b p_D(t - s, z, y) dz ds \quad \text{for } x, y \in D. \quad (1.7)$$

However, it is difficult to get pointwise estimate on $\mathcal{S}_z^b p_D(t - s, z, y)$. Following the general strategy developed in [10], we first derive sharp estimates on the Green function $G_D^b(x, y)$ of $X^{b, D}$.

The the Green function $G_D^b(x, y)$ on a bounded open set D satisfies the following Duhamel's formula:

$$G_D^b(x, y) = G_D(x, y) + \int_D G_D(x, z) \mathcal{S}_z^b G_D(z, y) dz \quad \text{for } x, y \in D, \quad (1.8)$$

where $G_D(x, y)$ is the Green function of the killed symmetric α -stable process $X^{0,D}$ in D . Applying the above formula recursively, one expects that $G_D^b(x, y)$ can be expressed as an infinite series in terms of $G_D(x, y)$ and $\mathcal{S}_z^b G_D(x, y)$. The main challenge is to derive sharp bound on $\mathcal{S}_z^b G_D(x, y)$ and to deduce from that $G_D^b(x, y)$ is comparable to $G_D(x, y)$ for $C^{1,1}$ open sets D having small diameter. From this, we can get the boundary decay rate of $p_D^b(t, x, y)$ and furthermore its sharp two-sided estimates. Integrating the two-sided estimates on $p_D^b(t, x, y)$, we can get two-sided sharp bound on $G_D^b(x, y)$ for any bounded $C^{1,1}$ open set D .

The rest of this paper is organized as follows. In Section 2, we review some known estimates for the global heat kernel $p^b(t, x, y)$ of X^b and some basic properties of a bounded $C^{1,1}$ open set. In Section 3 we derive some lower bound estimates for $p_D^b(t, x, y)$ that will be used later in this paper. Section 4 is devoted to the sharp two-sided estimates for Green functions of X^b in $C^{1,1}$ open sets with sufficiently small diameter. This is done through a series of lemmas, which provide proper estimates on $\mathcal{S}_z^b G_D(x, y)$. In Section 5 and Section 6 we obtain small time two-sided Dirichlet heat kernel estimates for $p_D^b(t, x, y)$. Large time estimates of $p_D^b(t, x, y)$ is obtained in Section 7 for bounded $C^{1,1}$ open sets.

2 Preliminaries

We first recall some estimates on the heat kernel $p^b(t, x, y)$ of \mathcal{L}^b from [12].

Proposition 2.1. *For every $A, \lambda \in (0, \infty)$, there are positive constants $C_k = C_k(d, \alpha, \beta, A, \lambda)$, $k = 7, \dots, 10$ such that for every bounded function b satisfying condition (1.2) and (1.4) with $\|b\|_\infty \leq A$, we have for every $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$C_7^{-1} \bar{p}_1(t, C_8 x, C_8 y) \leq p^b(t, x, y) \leq C_7 p_{M_{b^+, \lambda}}(t, x, y), \quad (2.1)$$

and for every $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_9^{-1} e^{-C_{10}t} \bar{p}_1(t, x, y) \leq p^b(t, x, y) \leq C_9 e^{C_{10}t} p_{M_{b^+, \lambda}}(t, x, y). \quad (2.2)$$

Here $M_{b, \lambda} := \text{esssup}_{x, z \in \mathbb{R}^d, |z| > \lambda} |b(x, z)|$. Define $m_{b, \lambda} := \text{essinf}_{x, z \in \mathbb{R}^d, |z| > \lambda} b(x, z)$. If b also satisfies (1.6) for some positive constant ε , then there are constants $C_k = C_k(d, \alpha, \beta, A, \varepsilon) \geq 1$, $k = 11, 12, 13$ that for every $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_{11}^{-1} p_{m_{b^+, \lambda}}(t, x, y) \leq p^b(t, x, y) \leq C_{11} p_{M_{b^+, \lambda}}(t, x, y), \quad (2.3)$$

and for every $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$C_{12}^{-1} e^{-C_{13}t} p_{m_{b^+, \lambda}}(t, x, y) \leq p^b(t, x, y) \leq C_{12} e^{C_{13}t} p_{M_{b^+, \lambda}}(t, x, y). \quad (2.4)$$

We will need the following known geometric properties of a $C^{1,1}$ open set D with $C^{1,1}$ characteristic (R_0, Λ_0) :

- (i) (outer and inner ball property) There is a constant $0 < r_0 = r_0(D) < \infty$ such that for any $Q \in \partial D$, $0 < r < r_0$, there are balls $B(x', r) \subset D$, $B(x'', r) \subset D^c$ tangent at Q . We also say that D is a $C^{1,1}$ open set at scale r_0 .
- (ii) There exists $L = L(d, R_0, \Lambda_0) > 0$ such that for every $z \in \partial D$, $0 < r \leq R_0$, one can find a $C^{1,1}$ open domain V with characteristic $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(z, r) \subset V \subset D \cap B(z, 2r)$. We will write $V = V(z, r)$.
- (iii) There exists a constant $\kappa = \kappa(\Lambda_0) \in (0, 1/2)$ such that for every $r \in (0, R_0)$ and $Q \in \partial D$, there is a point A in $D \cap B(Q, r)$, denoted by $A_r(Q)$, such that $B(A, \kappa r) \subset D \cap B(Q, r)$. (R_0, κ) is called the κ -fat characteristic of D .

Proposition 2.2. *Suppose D is a bounded $C^{1,1}$ open set with characteristic (R_0, Λ_0) . Then there is $\theta_0 = \theta_0(\Lambda_0) \in (0, 1)$ such that for every $x \in D$ and $r \in (0, R_0]$, there exists a ball $B(A, \theta_0 r) \subset D \cap B(x, r)$.*

Proof. It is known that there is $\kappa = \kappa(\Lambda_0) \in (0, 1/2)$ such that for every $Q \in \partial D$, there is $B(A, \kappa r) \subset D \cap B(Q, r)$.

Fix $x \in D$. If $\delta_D(x) > \kappa R_0$, then the assertion is true since $B(x, \kappa R_0) \subset D$. If $\delta_D(x) \leq \kappa R_0$, let $D_1 := \{y \in D : \delta_D(y) > \delta_D(x)\}$. Obviously D_1 is bounded $C^{1,1}$ open with characteristic $\Lambda_0(D_1) = \Lambda_0$ and $R_0(D_1) \geq (1 - 2\kappa)R_0$. Note that $x \in \partial D_1$. Thus for every $r \in (0, (1 - 2\kappa)R_0)$, there exists a ball $B(A_1, \kappa r) \subset D_1 \cap B(x, r) \subset D \cap B(x, r)$. In this case we conclude the assertion by setting $\theta_0 = \kappa(1 - 2\kappa)$. \square

The Feller process X^b has the Lévy system $(j^b(x, y)dy, t)$. Recall that the Lévy system $(j^b(x, y)dy, t)$ describes the jumps of the process X^b : for every non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, y, y) : s \geq 0, y \in \mathbb{R}^d\}$, every $x \in \mathbb{R}^d$ and stopping time T (with respect to the minimal admissible filtration of X^b),

$$\mathbb{E}_x \left[\sum_{s < T} f(s, X_{s-}^b, X_s^b) \right] = \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}^d} f(s, X_s^b, y) j^b(X_s^b, y) dy ds \right]. \quad (2.5)$$

3 Properties of subprocess

In this section, b is a bounded function satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A < \infty$ and X^b is the corresponding Feller process. Let $D \subset \mathbb{R}^d$ be an open subset. In this section, we study some basic properties of the subprocess $X^{b,D}$ of X^b killed upon leaving D . Recall that ∂ is a cemetery added to D . Let $D_\partial := D \cup \{\partial\}$. Define for every $x, y \in D$,

$$N^D(x, dy) := j^b(x, y)dy, \quad N^D(x, \partial) = \int_{D^c} j^b(x, y)dy.$$

It follows from (2.5) that $(N_D, t \wedge \tau_D^b)$ is a Lévy system for $X^{b,D}$, i.e., for any $x \in D$, any non-negative measurable function f on $[0, \infty) \times D \times D_\partial$ vanishing on $\{(s, x, y) \in [0, \infty) \times D \times D_\partial : x = y\}$ and stopping time T (with respect to the filtration of $X^{b,D}$),

$$\mathbb{E}_x \left[\sum_{t \leq T} f(s, X_{s-}^{b,D}, X_s^{b,D}) \right] = \mathbb{E}_x \left[\int_0^T \int_{D_\partial} f(s, X_s^{b,D}, y) N^D(X_s^{b,D}, dy) ds \right]. \quad (3.1)$$

Define

$$\varepsilon(A) := \left(\frac{1}{2A} \frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)} \right)^{1/(\alpha-\beta)}. \quad (3.2)$$

By the assumptions of b , we have for every $x \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$

$$\begin{aligned} j^b(x, y) &= \frac{\mathcal{A}(d, -\alpha)}{|x-y|^{d+\alpha}} + \mathcal{A}(d, -\beta) \frac{b(x, y-x)}{|x-y|^{d+\beta}} 1_{\{|x-y| < \varepsilon(A)\}} + \mathcal{A}(d, -\beta) \frac{b(x, y-x)}{|x-y|^{d+\beta}} 1_{\{|x-y| \geq \varepsilon(A)\}} \\ &\geq \frac{\mathcal{A}(d, -\alpha)}{|x-y|^{d+\alpha}} - \mathcal{A}(d, -\beta) \frac{A}{|x-y|^{d+\beta}} 1_{\{|x-y| < \varepsilon(A)\}} - \mathcal{A}(d, -\alpha) \frac{1}{|x-y|^{d+\alpha}} 1_{\{|x-y| \geq \varepsilon(A)\}} \\ &= \frac{\mathcal{A}(d, -\alpha)}{|x-y|^{d+\alpha}} 1_{\{|x-y| < \varepsilon(A)\}} - \mathcal{A}(d, -\beta) \frac{A}{|x-y|^{d+\beta}} 1_{\{|x-y| < \varepsilon(A)\}} \\ &= \frac{\mathcal{A}(d, -\alpha)}{|x-y|^{d+\alpha}} 1_{\{|x-y| < \varepsilon(A)\}} \left(1 - \frac{\mathcal{A}(d, -\beta)}{\mathcal{A}(d, -\alpha)} A |x-y|^{\alpha-\beta} \right) \\ &\geq \frac{1}{2} \bar{j}_{\varepsilon(A)}(x, y), \end{aligned} \quad (3.3)$$

where $\bar{j}_{\varepsilon(A)}(x, y) := \mathcal{A}(d, -\alpha) |x-y|^{-d-\alpha} 1_{\{|x-y| < \varepsilon(A)\}}$. In other words, we have for every $x \in \mathbb{R}^d$,

$$j^b(x, y) \geq \frac{1}{2} j(x, y) \quad \text{a.e. on } \{y \in \mathbb{R}^d : |x-y| < \varepsilon(A)\}. \quad (3.4)$$

Lemma 3.1. For any $\delta > 0$,

$$\lim_{s \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau_{B(x, \delta)}^b \leq s) = 0.$$

Proof. For every $x \in \mathbb{R}^d$, we have

$$\begin{aligned} &\mathbb{P}_x(\tau_{B(x, \delta)}^b \leq s) \\ &\leq \mathbb{P}_x(\tau_{B(x, \delta)}^b \leq s, X_s^b \in B(x, \delta/2)) + \mathbb{P}_x(X_s^b \in B(x, \delta/2)^c) \\ &\leq \mathbb{E}_x \left[P_{X_{\tau_{B(x, \delta)}^b}^b} \left(|X_{s-\tau_{B(x, \delta)}^b}^b - X_0^b| \geq \delta/2 \right) : \tau_{B(x, \delta)}^b \leq s \right] + \mathbb{P}_x(|X_s^b - X_0^b| \geq \delta/2) \\ &\leq 2 \sup_{t \in [0, s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z(|X_t^b - X_0^b| \geq \delta/2). \end{aligned}$$

Note that by (2.2), we have

$$\begin{aligned}
& \sup_{t \in [0, s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z \left(|X_t^b - X_0^b| \geq \delta/2 \right) \\
&= \sup_{t \in [0, s]} \sup_{z \in \mathbb{R}^d} \mathbb{P}_z \left(|X_t^b - z| \geq \delta/2 \right) \\
&= \sup_{t \in [0, s]} \sup_{z \in \mathbb{R}^d} \int_{|y-z| \geq \delta/2} p^b(t, z, y) dy \\
&\leq \sup_{t \in [0, s]} \sup_{z \in \mathbb{R}^d} c_1 e^{c_2 t} \int_{|y-z| \geq \delta/2} t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left(\frac{t}{|z-y|^{d+\alpha}} + \frac{t}{|z-y|^{d+\beta}} \right) dy \\
&\leq c_1 e^{c_2 s} \int_{\delta/2}^{\infty} (r^{-\alpha+1} + r^{-\beta+1}) dr \rightarrow 0, \text{ as } s \downarrow 0
\end{aligned}$$

for some constants $c_i = c_i(d, \alpha, \beta, A) > 0$, $i = 1, 2$. This proves the assertion. \square

Theorem 3.2. *Let D be an open set in \mathbb{R}^d . The density function $p_D^b(t, x, y)$ is jointly continuous in $(0, \infty) \times D \times D$ and satisfies*

$$p_D^b(t + s, x, y) = \int_D p_D^b(t, x, z) p_D^b(s, z, y) dz, \quad \forall t, s > 0.$$

Proof. By (1.5), we only need to show that $k_D^b(t, x, y) := \mathbb{E}_x \left[p^b(t - \tau_D^b, X_{\tau_D^b}^b, y) : \tau_D^b < t \right]$ is jointly continuous on $(0, \infty) \times D \times D$. By (2.2), there are positive constants $c_i = c_i(d, \alpha, \beta, A)$, $i = 1, \dots, 4$ such that

$$\begin{aligned}
p^b(t, x, y) &\leq c_1 e^{c_2 t} p_{\|b\|_\infty}(t, x, y) \\
&\leq c_3 e^{c_2 t} \left[t^{-d/\alpha} \wedge (\|b\|_\infty t)^{-d/\beta} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{\|b\|_\infty t}{|x-y|^{d+\beta}} \right) \right] \\
&\leq c_4 e^{c_2 t} \left[t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}} \right) \right].
\end{aligned}$$

Thus for any $t_0 > 0$ and $\delta > 0$, we have

$$\begin{aligned}
& \sup_{t \leq t_0} \sup_{|x-y| \geq \delta} p^b(t, x, y) \\
&\leq c_4 e^{c_2 t_0} \sup_{t \leq t_0, |x-y| \geq \delta} \left[t^{-d/\alpha} \wedge t^{-d/\beta} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}} \right) \right] \\
&\leq c_4 e^{c_2 t_0} \left(\frac{t_0}{\delta^{d+\alpha}} + \frac{t_0}{\delta^{d+\beta}} \right) =: c_5(d, \alpha, \beta, A, t_0, \delta) < \infty.
\end{aligned} \tag{3.5}$$

The assertion follows from Lemma 3.1 and (3.5) (instead of Lemma 3.1 and (3.6) in [10]) in the same way as for the case of fractional Laplacian with gradient perturbation in Theorem 3.4 of [10]. We omit the details here. \square

Lemma 3.3. *For any $a_1, \kappa_1 \in (0, 1)$, $R \in (0, 1/2]$ and $A > 0$, there are constants $l = l(d, \alpha, \beta, a_1, \kappa_1, R, A) \in (0, 1)$ and $C_{14} = C_{14}(d, \alpha, \beta, a_1, \kappa_1, R, A) > 0$ such that for any bounded function b satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$, any $x_0 \in \mathbb{R}^d$ and $r \in (0, R]$, we have*

$$p_{B(x_0, r)}^b(t, x, y) \geq C_{14} r^{-d} \quad \text{for } (t, x, y) \in [\kappa_1 l r^\alpha, l r^\alpha] \times B(x_0, a_1 r) \times B(x_0, a_1 r). \tag{3.6}$$

Moreover, if b also satisfies (1.6) for some $\varepsilon > 0$, then the above estimate holds for all $R > 0$ and some positive constants $l = l(d, \alpha, \beta, a_1, \kappa_1, R, A, \varepsilon)$ and $C_{14} = C_{14}(d, \alpha, \beta, a_1, \kappa_1, R, A, \varepsilon)$.

Proof. Fix $x_0 \in \mathbb{R}^d$. We use B_r to denote $B(x_0, r)$. Note that by (2.2),

$$\begin{aligned} & p_{B_r}^b(t, x, y) \\ &= p^b(t, x, y) - \mathbb{E}_x \left[p^b(t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) : \tau_{B_r}^b < t \right] \\ &\geq c_1 e^{-c_2 t} \bar{p}_1(t, x, y) - \mathbb{E}_x \left[c_3 e^{c_4(t - \tau_{B_r}^b)} p_{\|b\|_\infty}(t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) : \tau_{B_r}^b < t \right]. \end{aligned}$$

For every $x, y \in B(x_0, a_1 r)$ with $r \leq 1/2$ and $a_1 \in (0, 1)$, we have $|x - y| \leq 2a_1 r < 1$, and thus for any $t \in [\kappa_1 l r^\alpha, l r^\alpha] \subset (0, 1]$,

$$\begin{aligned} \bar{p}_1(t, x, y) &\geq c_5(d, \alpha) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \\ &\gtrsim c_6(d, \alpha, a_1) t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{r} \right)^{d+\alpha} \\ &\geq c_6(l r^\alpha)^{-d/\alpha} \left(1 \wedge \frac{(\kappa_1 l)^{1/\alpha} r}{r} \right)^{d+\alpha} \\ &= c_6 \kappa_1^{1+d/\alpha} l r^{-d}. \end{aligned} \tag{3.7}$$

On the other hand, since $|X_{\tau_{B_r}^b}^b - y| \geq (1 - a_1)r$ for every $y \in B(x_0, a_1 r)$, we have

$$\begin{aligned} & p_{\|b\|_\infty}(t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) \\ &\leq c_7(d, \alpha, \beta) \left[(t - \tau_{B_r}^b)^{-d/\alpha} \wedge (\|b\|_\infty(t - \tau_{B_r}^b))^{-d/\beta} \wedge \left(\frac{t}{|X_{\tau_{B_r}^b}^b - y|^{d+\alpha}} + \frac{\|b\|_\infty t}{|X_{\tau_{B_r}^b}^b - y|^{d+\beta}} \right) \right] \\ &\leq c_8(d, \alpha, \beta, A, a_1) \left(\frac{t}{r^{d+\alpha}} + \frac{t}{r^{d+\beta}} \right) \\ &\leq c_9(d, \alpha, \beta, A, a_1, R) \frac{t}{r^{d+\alpha}}. \end{aligned} \tag{3.8}$$

It follows from the proof of Lemma 3.1 that for every $x \in B(x_0, a_1 r)$,

$$\mathbb{P}_x(\tau_{B_r}^b < t) \leq \mathbb{P}_x(\tau_{B(x, (1-a_1)r)}^b \leq t) \leq 2 \sup_{s \in [0, t], z \in \mathbb{R}^d} \mathbb{P}_z(X_s^b \notin B(z, (1-a_1)r/2)) \tag{3.9}$$

where in the last inequality we have for every $s \in (0, t]$ and $z \in \mathbb{R}^d$,

$$\begin{aligned} & \mathbb{P}_z(X_s^b \notin B(z, (1-a_1)r/2)) \\ &= \int_{B(z, (1-a_1)r/2)^c} p^b(s, z, y) dy \\ &\leq c_{10} e^{c_4 s}(d, \alpha, \beta, A) \int_{B(z, (1-a_1)r/2)^c} s \left(|z - y|^{-d-\alpha} + |z - y|^{-d-\beta} \right) ds \\ &\leq c_{11} e^{c_4 t}(d, \alpha, \beta, A, a_1) t (r^{-d-\alpha} + r^{-d-\beta}) \\ &\leq c_{12} e^{c_4 t}(d, \alpha, \beta, A, a_1, R) \frac{t}{r^\alpha}. \end{aligned} \tag{3.10}$$

Thus by (3.8) (3.9) and (3.10), we get for every $t \in [\kappa_1 l r^\alpha, l r^\alpha]$ and $x, y \in B(x_0, a_1 r)$,

$$\begin{aligned} & \mathbb{E}_x \left[c_3 e^{c_4(t-\tau_{B_r}^b)} p_{\|b\|_\infty}(t - \tau_{B_r}^b, X_{\tau_{B_r}^b}^b, y) : \tau_{B_r}^b < t \right] \\ & \leq c_{13}(d, \alpha, \beta, A, a_1, R) e^{2c_4 t} \frac{t^2}{r^{d+2\alpha}} \\ & \leq c_{13} e^{2c_4 l R^\alpha} l^2 r^{-d}. \end{aligned} \quad (3.11)$$

Therefore, by (3.7) and (3.11) we have

$$p_{B_r}^b(t, x, y) \geq l e^{-c_2 l R^\alpha} (c_1 c_6 \kappa_1^{1+d/\alpha} - c_{13} l e^{(2c_4+c_2)l R^\alpha}) r^{-d}.$$

The first assertion of Lemma 3.3 follows by setting $l = l(d, \alpha, \beta, a_1, \kappa_1, R, A)$ sufficiently small such that $(c_1 c_6 \kappa_1^{1+d/\alpha} - c_{13} l e^{(2c_4+c_2)l R^\alpha}) > 0$. Moreover, if b also satisfies (1.6), then by Proposition 2.1, we have for every $t \in (0, \infty)$ and every $x, y \in B_r$ with $0 < r < \infty$,

$$p_{B_r}^b(t, x, y) \geq c_{14} e^{-c_{15} t} p(t, x, y) - \mathbb{E}_x \left[c_{16} e^{c_{16}(t-\tau_{B(r)}^b)} p_{\|b\|_\infty}(t - \tau_{B(r)}^b, X_{\tau_{B(r)}^b}^b, y) : \tau_{B(r)}^b < t \right].$$

Using the estimate that $p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$, one can deduce by a similar argument as above that estimates (3.6) holds for $r \in (0, R]$ for all $R > 0$. \square

Proposition 3.4. *For any $a_1 \in (0, 1)$, $a_3 > a_2 > 0$, $R \in (0, 1/2]$ and $A > 0$, there is a positive constant $C_{15} = C_{15}(d, \alpha, \beta, a_1, a_2, a_3, R, A)$ such that for every bounded function b satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$, every $x_0 \in \mathbb{R}^d$ and $r \in (0, R]$, we have*

$$p_{B(x_0, r)}^b(t, x, y) \geq C_{15} r^{-d} \quad \text{for every } t \in [a_2 r^\alpha, a_3 r^\alpha], \quad x, y \in B(x_0, a_1 r).$$

Moreover, if b also satisfies the condition (1.6) for some $\varepsilon > 0$, then the above estimate holds for all $R > 0$ and some $C_{15} = C_{15}(d, \alpha, \beta, a_1, a_2, a_3, R, A, \varepsilon) > 0$.

Proof. We can choose appropriate $\kappa_1 \in (0, 1)$ and $k \in \mathbb{N}$ such that $a_3/l \leq k \leq a_2/(\kappa_1 l)$ where $l = l(d, \alpha, \beta, a_1, \kappa_1, R, A) \in (0, 1)$ is the constant defined in Lemma 3.3. In this case $t/k \in [\kappa_1 l r^\alpha, l r^\alpha]$ for every $t \in [a_2 r^\alpha, a_3 r^\alpha]$. Thus by semigroup property and Lemma 3.3, we have

$$\begin{aligned} & p_{B(x_0, r)}^b(t, x, y) \\ &= \int_{B(x_0, r)} \cdots \int_{B(x_0, r)} p_{B(x_0, r)}^b(t/k, x, z_1) \cdots p_{B(x_0, r)}^b(t/k, z_{k-1}, y) dz_1 \cdots dz_{k-1} \\ &\geq (C_{14} r^{-d})^k m(B(x_0, r))^{k-1} \geq c_1 r^{-d} \end{aligned}$$

for some $c_1 = c_1(d, \alpha, \beta, a_1, a_2, a_3, R, A) > 0$. \square

Lemma 3.5. *Suppose D is an open set in \mathbb{R}^d . For every $x \in D$, we use D_x to denote the connected component of D that contains x . Then $p_D^b(t, x, y) > 0$ for every $t > 0$ and $x, y \in D$ with $\text{dist}(D_x, D_y) < \varepsilon(A)$.*

Proof. Fix $x, y \in D$. If $y \in D_x$, then the assertion follows from the domain monotonicity of p_D^b , a chain argument and Proposition 3.4. If $y \notin D_x$, then by the strong Markov property, (2.5) and (3.4), we have

$$\begin{aligned}
& p_D^b(t, x, y) \\
&= \mathbb{E}_x \left[p_D^b(t - \tau_{D_x}^b, X_{\tau_{D_x}^b}^b, y) : \tau_{D_x}^b < t \right] \\
&\geq \mathbb{E}_x \left[p_D^b(t - \tau_{D_x}^b, X_{\tau_{D_x}^b}^b, y) : X_{\tau_{D_x}^b}^b \in D_y, \tau_{D_x}^b < t \right] \\
&= \int_0^t \int_{D_y} p_D^b(t - s, z, y) \left[\int_{D_x} p_{D_x}^b(s, x, w) j^b(w, z) dw \right] dz ds \\
&\geq \int_0^t \int_{D_y} \int_{D_x} p_{D_y}^b(t - s, z, y) p_{D_x}^b(s, x, w) j^b(w, z) dw dz ds \\
&\geq \frac{1}{2} \int_0^t \int_{D_y} \int_{\{w \in D_x : \text{dist}(w, D_y) < \varepsilon(A)\}} p_{D_y}^b(t - s, z, y) p_{D_x}^b(s, x, w) \bar{j}_{\varepsilon(A)}(w, z) dw dz ds \\
&> 0.
\end{aligned}$$

□

4 Green function estimates

Suppose D is a bounded open set. Let $G_D^b(x, y)$ denote the Green function of the subprocess $X^{b,D}$. For any $\lambda > 0$, define

$$b_\lambda(x, z) := \lambda^{\beta-\alpha} b(\lambda^{-1}x, \lambda^{-1}z), \quad x, z \in \mathbb{R}^d. \quad (4.1)$$

Obviously if $\|b\|_\infty \leq A$ then $\|b_\lambda\|_\infty \leq \lambda^{\beta-\alpha} A =: A_\lambda$. Hereafter, we call a constant c depending on D, b and A (part of them) *scale-invariant* if it satisfies $c(\lambda D, b_\lambda, A_\lambda) = c(D, b, A)$.

It is not hard to prove that $\lambda X_{\lambda^{-\alpha}t}^b$ has the same distribution as $X_t^{b_\lambda}$, while for any open set D , $\lambda X_{\lambda^{-\alpha}t}^{b,D}$ has the same distribution as $X_t^{b_\lambda, \lambda D}$. So for any $\lambda > 0$, we have the following scaling properties:

$$p^b(t, x, y) = \lambda^d p^{b_\lambda}(\lambda^\alpha t, \lambda x, \lambda y), \quad x, y \in \mathbb{R}^d, \quad t > 0. \quad (4.2)$$

$$p_D^b(t, x, y) = \lambda^d p_{\lambda D}^{b_\lambda}(\lambda^\alpha t, \lambda x, \lambda y), \quad x, y \in D, \quad t > 0. \quad (4.3)$$

$$G_D^b(x, y) = \lambda^{d-\alpha} G_{\lambda D}^{b_\lambda}(\lambda x, \lambda y), \quad x, y \in D. \quad (4.4)$$

Suppose X is a symmetric α -stable process. We will use τ_D to denote the first time that X exits D . Let $G(x, y), G_D(x, y)$ and $K_D(x, y)$ denote respectively the global Green function of X , the Green function and Poisson kernel of subprocess X killed upon exiting D . Let $B(x_0, r)$ be an arbitrary ball in \mathbb{R}^d . The explicit formulas for $G(x, y), G_{B(x_0, r)}(x, y)$ and $K_{B(x_0, r)}(x, y)$ are known as follows: For every $x, y \in \mathbb{R}^d$,

$$G(x, y) = 2^{-\alpha} \pi^{-d/2} \Gamma\left(\frac{d-\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-1} |x - y|^{\alpha-d}. \quad (4.5)$$

For every $x, y \in B(x_0, r)$,

$$G_{B(x_0, r)}(x, y) = 2^{-\alpha} \pi^{-d/2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-2} \int_0^z (u+1)^{-d/2} u^{\alpha/2-1} du |x-y|^{\alpha-d}, \quad (4.6)$$

where $z = (r^2 - |x - x_0|^2)(r^2 - |y - x_0|^2)|x - y|^{-2}$. For every $x \in B(x_0, r)$ and $y \in \overline{B(x_0, r)}^c$,

$$K_{B(x_0, r)}(x, y) = c(d, \alpha) (r^2 - |x - x_0|^2)^{\alpha/2} (|y - x_0|^2 - r^2)^{-\alpha/2} |x - y|^{-d}. \quad (4.7)$$

where $c(d, \alpha) = \Gamma(\frac{d}{2}) \sin \frac{\pi\alpha}{2} \pi^{-d/2-1}$. It is known that (see [11, 14]) for any bounded $C^{1,1}$ open set D with characteristic (R_0, Λ_0) , there exists a constant $c_0 = c_0(d, \alpha, D) > 1$ such that

$$G_D(x, y) \stackrel{c_0}{\asymp} |x - y|^{\alpha-d} \left(1 \wedge \frac{\delta_D(x)}{|x - y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} \quad x, y \in D. \quad (4.8)$$

Here $\delta_D(z) := \text{dist}(z, \partial D)$. It follows from the scaling property

$$G_D(x, y) = \lambda^{d-\alpha} G_{\lambda D}(\lambda x, \lambda y), \quad x, y \in D, \quad \lambda > 0 \quad (4.9)$$

that the constant c_0 can be chosen to be scale-invariant.

Definition 4.1. We say that function u defined on \mathbb{R}^d is \mathcal{L}^b -harmonic on an open set D if it satisfies

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_U^b}^b) \right] \quad (4.10)$$

for every bounded open set U with closure \bar{U} contained in D . It is called regular \mathcal{L}^b -harmonic if (4.10) holds for $U = D$.

Note that when D is unbounded, by the usual convention,

$$\mathbb{E}_x \left[u(X_{\tau_D^b}^b) \right] = \mathbb{E}_x \left[u(X_{\tau_D^b}^b); \tau_D^b < \infty \right].$$

It is always assumed that the expectation in (4.10) is absolutely convergent. In particular, $G_D^b(\cdot, y)$ is \mathcal{L}^b -harmonic in $D \setminus \{y\}$. Indeed, $G_D^b(x, y) = G_U^b(x, y) + \mathbb{E}_x G_D^b(X_{\tau_U^b}^b, y)$ for every open set $U \subset D$. We point out that in general $G_D^b(x, y) \neq G_D^b(y, x)$, and $G_D^b(x, \cdot)$ is not \mathcal{L}^b -harmonic. The definition of α -harmonicity for $\Delta^{\alpha/2}$ is analogous to that of \mathcal{L}^b -harmonicity.

Lemma 4.2. Suppose D is a bounded open set in \mathbb{R}^d and $A \in (0, \infty)$. For every $x \in D$, $y \mapsto G_D^b(x, y)$ is continuous in $D \setminus \{x\}$. Moreover, there exists a scale-invariant constant $C_{16} = C_{16}(d, \alpha, \beta, D, A) > 0$ such that for any bounded function b satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$,

$$G_D^b(x, y) \leq C_{16} |x - y|^{-d+\alpha}, \quad x, y \in D. \quad (4.11)$$

Proof. First we claim that there exist positive constants c_1 and c_2 depending on $d, \alpha, \beta, \text{diam}(D)$ and A such that for any $1 \leq t < \infty$, $x, y \in D$, and $\|b\|_\infty \leq A$

$$p_D^b(t, x, y) \leq c_1 e^{-c_2 t}. \quad (4.12)$$

This inequality follows from a standard argument using (2.1) and Markov property (see, for example [10] Lemma 3.7). Thus we have

$$\begin{aligned}
G_D^b(x, y) &= \int_0^\infty p_D^b(t, x, y) dt \\
&\leq \int_0^1 p^b(t, x, y) dt + \int_0^\infty p_D^b(t, x, y) dt \\
&\lesssim \int_0^1 t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{At}{|x-y|^{d+\beta}} \right) dt + \int_1^\infty c_1 e^{-c_2 t} dt \\
&\leq (1 + A|x-y|^{\alpha-\beta}) \int_0^1 t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} dt + c_1/c_2 \\
&\leq (1 + \text{Adiam}(D)^{\alpha-\beta}) |x-y|^{\alpha-d} + c_1/c_2 \\
&\leq \left[(1 + \text{Adiam}(D)^{\alpha-\beta}) + c_1 \text{diam}(D)^{d-\alpha}/c_2 \right] |x-y|^{\alpha-d}.
\end{aligned}$$

The scale-invariance of C_{16} is implied by (4.4). By (4.12), (2.1) and the dominated convergence theorem, $y \mapsto G_D^b(x, y)$ is continuous if $y \neq x$. \square

The first part of the next two lemmas is proved in Lemma 3.1 and Lemma 3.2 of [3], respectively, while the second inequality can be proved by a similar argument. Hence we omit their proofs.

Lemma 4.3. *There is a positive constant $C_{17} = C_{17}(d, \alpha)$ such that for any $r > 0$ and ball $B := B(0, r)$, we have*

$$|\nabla_x K_B(x, z)| \leq C_{17} \frac{K_B(x, z)}{\delta_B(x)}, \quad |\partial_{ij} K_B(x, z)| \leq C_{17} \frac{K_B(x, z)}{\delta_B(x)^2}, \quad \forall (x, z) \in B \times \bar{B}^c.$$

Here $\nabla_x := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ and $\partial_{i,j} := \frac{\partial^2}{\partial x_i \partial x_j}$.

Lemma 4.4. *There is a positive constant $C_{18} = C_{18}(d, \alpha)$ such that for an arbitrary open set D in \mathbb{R}^d , and every non-negative function f which is α -harmonic in D , we have*

$$|\nabla_x f(x)| \leq C_{18} \frac{f(x)}{\delta_D(x)}, \quad |\partial_{ij} f(x)| \leq C_{18} \frac{f(x)}{\delta_D(x)^2}, \quad \forall x \in D, \quad i, j \in \{1, \dots, d\}.$$

Lemma 4.5. *Let D be a $C^{1,1}$ open set in \mathbb{R}^d . There exists a scale-invariant constant $C_{19} = C_{19}(d, \alpha, D) > 0$ such that*

$$|\nabla_x G_D(x, y)| \leq C_{19} |x-y|^{\alpha-d-1} \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2} \left(1 \vee \frac{|x-y|}{\delta_D(x)} \right)^{1-\alpha/2}, \quad (4.13)$$

$$|\partial_{ij} G_D(x, y)| \leq C_{19} |x-y|^{\alpha-d-2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2} \left(1 \vee \frac{|x-y|}{\delta_D(x)} \right)^{2-\alpha/2} \quad (4.14)$$

for every $x, y \in D$.

Proof. For each $y \in D$ and $1 \leq i, j \leq d$, we have by Lemma 4.4 applied to domain $D \setminus \{y\}$,

$$|\nabla_x G_D(x, y)| \leq c_1 \frac{G_D(x, y)(x)}{|x - y| \wedge \delta_D(x)}, \quad |\partial_{ij} G_D(x, y)| \leq c_1 \frac{G_D(x, y)}{(|x - y| \wedge \delta_D(x))^2}, \quad x \in D \setminus \{y\}.$$

So it follows from (4.8) that

$$\begin{aligned} |\nabla_x G_D(x, y)| &\leq c_1 |x - y|^{\alpha-d-1} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(x)}{|x - y|}\right)^{\alpha/2} \left(1 \vee \frac{|x - y|}{\delta_D(x)}\right) \\ &\leq c_1 |x - y|^{\alpha-d-1} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} \left(1 \vee \frac{|x - y|}{\delta_D(x)}\right)^{1-\alpha/2}. \end{aligned}$$

The second derivative estimate on $G_D(x, y)$ is similar. \square

For $x \neq y$ in D , define

$$h_D(x, y) := \begin{cases} |x - y|^{\alpha-\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} & \text{if } \alpha > 2\beta, \\ |x - y|^{\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\beta} \left(1 \vee \log \frac{|x - y|}{\delta_D(x)}\right) & \text{if } \alpha = 2\beta, \\ |x - y|^{\alpha-\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} \left(1 \vee \frac{|x - y|}{\delta_D(x)}\right)^{\beta-\alpha/2} & \text{if } \alpha < 2\beta, \end{cases} \quad (4.15)$$

and

$$\begin{aligned} |\mathcal{S}_x^b| G_D(x, y) &:= \mathcal{A}(d, -\beta) \left(\int_{|z| \leq \lambda} |G_D(x + z, y) - G_D(x, y) - \nabla_x G_D(x, y) \cdot z| \frac{|b(x, z)|}{|z|^{d+\beta}} dz \right. \\ &\quad \left. + \int_{|z| > \lambda} (G_D(x + z, y) + G_D(x, y)) \frac{|b(x, z)|}{|z|^{d+\beta}} dz \right). \end{aligned} \quad (4.16)$$

where $\lambda := (\delta_D(x) \wedge |x - y|)/2 > 0$.

Lemma 4.6. *Let D be a bounded $C^{1,1}$ open set. Then there is a positive scale-invariant constant $C_{20} = C_{20}(d, \alpha, \beta, D)$ such that for every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$,*

$$|\mathcal{S}_x^b| G_D(x, y) \leq C_{20} \|b\|_{\infty} h_D(x, y). \quad (4.17)$$

Proof. Obviously we have

$$\begin{aligned} &|\mathcal{S}_x^b| G_D(x, y) \\ &\leq \mathcal{A}(d, -\beta) \|b\|_{\infty} \left(\int_{|z| \leq \lambda} |G_D(x + z, y) - G_D(x, y) - \nabla_x G_D(x, y) \cdot z| |z|^{-d-\beta} dz \right. \\ &\quad \left. + \int_{|z| > \lambda, x+z \in D} G_D(x + z, y) |z|^{-d-\beta} dz + \int_{|z| > \lambda} G_D(x, y) |z|^{-d-\beta} dz \right) \\ &=: \mathcal{A}(d, -\beta) \|b\|_{\infty} (I + II + III). \end{aligned}$$

Define $r_D(x, y) := \delta_D(x) + \delta_D(y) + |x - y|$. Since $\delta_D(y) \leq \delta_D(x) + |x - y|$, we have $\delta_D(x) + |x - y| \leq r_D(x, y) \leq 2(\delta_D(x) + |x - y|)$, in other words, we have $r_D(x, y) \asymp \delta_D(x) + |x - y| \asymp \delta_D(y) + |x - y|$. It is know that for every $a, b, p \geq 0$,

$$a \wedge b \asymp ab/(a + b), \quad a \vee b \asymp a + b, \quad a^p + b^p \asymp (a + b)^p. \quad (4.18)$$

Immediately we have $\lambda \asymp \delta_D(x)|x-y|/r_D(x,y)$. Using (4.18) repeatedly, we have

$$\begin{aligned}
III &\leq c_0 \beta^{-1} |x-y|^{\alpha-d} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \int_{|z|>\lambda} |z|^{-d-\beta} dz \\
&= c_0 \beta^{-1} |x-y|^{\alpha-d} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \lambda^{-\beta} \\
&\asymp c_0 \beta^{-1} |x-y|^{\alpha-d} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \frac{\delta_D(x)^{\alpha/2}}{r_D(x,y)^{\alpha/2}} \frac{r_D(x,y)^\beta}{\delta_D(x)^\beta |x-y|^\beta} \\
&\asymp c_0 \beta^{-1} |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(\frac{r_D(x,y)}{\delta_D(x)}\right)^{\beta-\alpha/2}.
\end{aligned} \tag{4.19}$$

Next we deal with I . Note that for $|z| \leq \lambda$, by (4.14),

$$\begin{aligned}
&|G_D(x+z, y) - G_D(x, y) - \nabla_x G_D(x, y) \cdot z| \\
&\leq \frac{1}{2} |z|^2 \sup_{|u| \leq \lambda} \sum_{1 \leq i, j \leq d} |\partial_{ij} G_D(x+u, y)| \\
&\leq \frac{1}{2} |z|^2 d^2 C_{19} \sup_{|u| \leq \lambda} |x+u-y|^{\alpha-d-2} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x+u-y|^{\alpha/2}}\right) \left(1 \vee \frac{|x+u-y|^{2-\alpha/2}}{\delta_D(x+u)^{2-\alpha/2}}\right).
\end{aligned} \tag{4.20}$$

It is easy to see that for every $|u| \leq \lambda = \frac{1}{2}(\delta_D(x) \wedge |x-y|)$, we have $|x-y|/2 \leq |x+u-y| \leq 3|x-y|/2$ and $\delta_D(x+u) \geq \delta_D(x) - |u| \geq \delta_D(x)/2$, thus

$$(4.20) \lesssim \frac{1}{2} |z|^2 d^2 C_{19} |x-y|^{\alpha-d-2} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \vee \frac{|x-y|^{2-\alpha/2}}{\delta_D(x)^{2-\alpha/2}}\right),$$

and consequently,

$$\begin{aligned}
I &\lesssim \frac{1}{2} d^2 C_{19} (2-\beta)^{-1} |x-y|^{\alpha-d-2} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \vee \frac{|x-y|^{2-\alpha/2}}{\delta_D(x)^{2-\alpha/2}}\right) \int_{|z| \leq \lambda} |z|^{2-d-\beta} dz \\
&= \frac{1}{2} d^2 C_{19} (2-\beta)^{-1} |x-y|^{\alpha-d-2} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \vee \frac{|x-y|^{2-\alpha/2}}{\delta_D(x)^{2-\alpha/2}}\right) \lambda^{2-\beta} \\
&\asymp \frac{1}{2} d^2 C_{19} (2-\beta)^{-1} |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(\frac{r_D(x,y)}{\delta_D(x)}\right)^{\beta-\alpha/2}.
\end{aligned} \tag{4.21}$$

Now we deal with II .

$$\begin{aligned}
II &\leq c_0 \int_{\substack{x+z \in D \\ |z| > \lambda}} |x+z-y|^{-d+\alpha} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x+z-y|^{\alpha/2}}\right) \left(1 \wedge \frac{\delta_D(x+z)^{\alpha/2}}{|x+z-y|^{\alpha/2}}\right) |z|^{-d-\beta} dz \\
&= c_0 \left(\int_{\substack{x+z \in D \\ \lambda < |z| < 3|x-y|/4}} + \int_{\substack{x+z \in D \\ |z| \geq 3|x-y|/4}} \right) \cdots dz \\
&=: c_0 (IV + V).
\end{aligned}$$

As for IV , we observe that if $\lambda < |z| < 3|x-y|/4$, we have $\frac{1}{4}|x-y| \leq |x+z-y| \leq 7|x-y|/4$. By this and (4.18), we get

$$\begin{aligned}
IV &\asymp |x-y|^{-d+\alpha} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \int_{\substack{x+z \in D \\ \lambda < |z| < 3|x-y|/4}} \left(1 \wedge \frac{\delta_D(x+z)^{\alpha/2}}{|x+z-y|^{\alpha/2}}\right) |z|^{-d-\beta} dz \\
&\asymp |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \int_{\substack{x+z \in D \\ \lambda < |z| < 3|x-y|/4}} |x-y|^\beta \frac{\delta_D(x+z)^{\alpha/2}}{(\delta_D(x+z) + |x-y|)^{\alpha/2}} |z|^{-d-\beta} dz
\end{aligned} \tag{4.22}$$

We continue to estimate the integral in (4.22). If $\delta_D(x) \geq |x-y|$, then $\lambda = |x-y|/2$, and

$$\begin{aligned}
&\int_{\substack{x+z \in D \\ |x-y|/2 < |z| < 3|x-y|/4}} |x-y|^\beta \frac{\delta_D(x+z)^{\alpha/2}}{(\delta_D(x+z) + |x-y|)^{\alpha/2}} |z|^{-d-\beta} dz \\
&\leq \int_{\frac{1}{2}|x-y| < |z| < \frac{3}{4}|x-y|} |x-y|^\beta |z|^{-d-\beta} dz \\
&= \int_{1/2}^{3/4} r^{-\beta-1} dr < \infty.
\end{aligned}$$

Consequently, we have for $\delta_D(x) \geq |x-y|$,

$$IV \lesssim |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right). \tag{4.23}$$

Otherwise if $\delta_D(x) < |x-y|$, then $\lambda = \delta_D(x)/2$. Note that $\delta_D(x+z) \leq \delta_D(x) + |z| \leq 3|z|$ for any z satisfying $\frac{1}{2}\delta_D(x) < |z| < \frac{3}{4}|x-y|$.

When $\alpha > 2\beta$, we have

$$\begin{aligned}
&\int_{\substack{x+z \in D \\ \delta_D(x)/2 < |z| < 3|x-y|/4}} |x-y|^\beta \frac{\delta_D(x+z)^{\alpha/2}}{(\delta_D(x+z) + |x-y|)^{\alpha/2}} |z|^{-d-\beta} dz \\
&= \int_{\substack{x+z \in D \\ \delta_D(x)/2 < |z| < 3|x-y|/4}} |x-y|^{\beta-\alpha/2} |z|^{-d+\alpha/2-\beta} \frac{|x-y|^{\alpha/2}}{(\delta_D(x+z) + |x-y|)^{\alpha/2}} \frac{\delta_D(x+z)^{\alpha/2}}{\delta_D(z)^{\alpha/2}} dz \\
&\lesssim \int_{\delta_D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta-\alpha/2} |z|^{-d+(\alpha/2-\beta)} dz \\
&= \int_{\delta_D(x)/(2|x-y|) \leq |u| \leq 3/4} |u|^{-d+(\alpha/2-\beta)} du \\
&\leq \int_0^{3/4} r^{\alpha/2-\beta-1} dr < \infty.
\end{aligned} \tag{4.24}$$

When $\alpha = 2\beta$, we have

$$\begin{aligned}
& \int_{\delta_D(x)/2 < |z| < 3|x-y|/4}^{x+z \in D} |x-y|^\beta \frac{\delta_D(x+z)^\beta}{(\delta_D(x+z) + |x-y|)^\beta} |z|^{-d-\beta} dz \\
& \leq \int_{\delta_D(x)/2 < |z| < 3|x-y|/4}^{x+z \in D} (3|z|)^\beta |z|^{-d-\beta} dz \\
& \asymp \int_{1/2 < |u| < 3|x-y|/(4\delta_D(x))} |u|^{-d} du \\
& \lesssim \log \frac{|x-y|}{\delta_D(x)}.
\end{aligned} \tag{4.25}$$

For $\alpha < 2\beta$, we have

$$\begin{aligned}
& \int_{\delta_D(x)/2 < |z| < 3|x-y|/4}^{x+z \in D} |x-y|^\beta \frac{\delta_D(x+z)^{\alpha/2}}{(\delta_D(x+z) + |x-y|)^{\alpha/2}} |z|^{-d-\beta} dz \\
& \lesssim \int_{\delta_D(x)/2 < |z| < 3|x-y|/4} |x-y|^{\beta-\alpha/2} |z|^{-d-(\beta-\alpha/2)} dz \\
& \leq \left(\frac{|x-y|}{\delta_D(x)} \right)^{\beta-\alpha/2} \int_{1/2 < |u| < 3|x-y|/(4\delta_D(x))} |u|^{-d-(\beta-\alpha/2)} du \\
& \leq \left(\frac{|x-y|}{\delta_D(x)} \right)^{\beta-\alpha/2} \int_{1/2}^{\infty} r^{-(\beta-\alpha/2)-1} dr \\
& \lesssim \left(\frac{|x-y|}{\delta_D(x)} \right)^{\beta-\alpha/2}.
\end{aligned} \tag{4.26}$$

We have from (4.23)-(4.26)

$$IV \leq c_1 |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \quad \text{when } \alpha > 2\beta. \tag{4.27}$$

$$IV \leq c_1 |x-y|^{-d+\beta} \left(1 \wedge \frac{\delta_D(y)^\beta}{|x-y|^\beta} \right) \left(1 \vee \log \frac{|x-y|}{\delta_D(x)} \right) \quad \text{when } \alpha = 2\beta, \tag{4.28}$$

and

$$IV \leq c_1 |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \left(1 \vee \frac{|x-y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \right) \quad \text{when } \alpha < 2\beta, \tag{4.29}$$

where $c_1 = c_1(d, \alpha, \beta) > 0$. As for V , note that

$$\begin{aligned}
V &= \int_{|u-x| \geq \frac{3}{4}|x-y|, u \in D} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) \left(1 \wedge \frac{\delta_D(u)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) du
\end{aligned} \tag{4.30}$$

Let $x' := x/|x - y|$ and $y' := y/|x - y|$. On one hand,

$$\begin{aligned}
V &\leq \int_{|u-x| \geq \frac{3}{4}|x-y|} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} du \\
&= |x-y|^{\alpha-d-\beta} \int_{|u'-x'| \geq 3/4} |u'-y'|^{-d+\alpha} |u'-x'|^{-d-\beta} du' \\
&= |x-y|^{\alpha-d-\beta} \int_{|v-(x'-y')| \geq 3/4} |v|^{-d+\alpha} (|v-(x'-y')| \vee \frac{3}{4})^{-d-\beta} dv \\
&\lesssim |x-y|^{\alpha-d-\beta} \int_{|v-(x'-y')| \geq 3/4} |v|^{-d+\alpha} (|v-(x'-y')| + \frac{3}{4})^{-d-\beta} dv \\
&\leq |x-y|^{\alpha-d-\beta} \int_{\mathbb{R}^d} |v|^{-d+\alpha} (|v| - 1 + \frac{3}{4})^{-d-\beta} dv \\
&\lesssim |x-y|^{\alpha-d-\beta}
\end{aligned} \tag{4.31}$$

On the other hand,

$$\begin{aligned}
V &\leq \int_{|u-x| \geq \frac{3}{4}|x-y|, u \in D} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|u-y|^{\alpha/2}} \right) du \\
&\asymp |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \int_{|u-x| \geq \frac{3}{4}|x-y|, u \in D} |x-y|^{d+\beta-\alpha/2} |u-y|^{-d+\alpha} |u-x|^{-d-\beta} r_D(u, y)^{-\alpha/2} du \\
&\leq |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \int_{|u-x| \geq \frac{3}{4}|x-y|, u \in D} |x-y|^{d+\beta-\alpha/2} |u-y|^{-d+\alpha/2} |u-x|^{-d-\beta} du \\
&\leq |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \int_{|u'-x'| \geq \frac{3}{4}} |u'-y'|^{-d+\alpha/2} |u'-x'|^{-d-\beta} du' \\
&\asymp |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \int_{|v-(x'-y')| \geq \frac{3}{4}} |v|^{-d+\alpha/2} (|v-(x'-y')| + \frac{3}{4})^{-d-\beta} dv \\
&\leq |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \int_{\mathbb{R}^d} |v|^{-d+\alpha/2} (|v| - 1 + \frac{3}{4})^{-d-\beta} dv \\
&\lesssim |x-y|^{\alpha-d-\beta} \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}.
\end{aligned} \tag{4.32}$$

Therefore by (4.31) and (4.32) we have

$$V \leq c_2 |x-y|^{\alpha-d-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \tag{4.33}$$

for some positive constant $c_2 = c_2(d, \alpha, \beta)$. Now we can complete the proof by combining (4.19), (4.21), (4.27), (4.28), (4.29) and (4.33), and using the fact that for $\alpha \geq 2\beta$,

$$\frac{r_D(x, y)^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp 1 \wedge \frac{\delta_D(x)^{\alpha/2-\beta}}{|x-y|^{\alpha/2-\beta}},$$

while for $\alpha < 2\beta$,

$$\frac{r_D(x, y)^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp 1 \vee \frac{|x-y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}}.$$

□

By (4.16) and Lemma 4.6, we have for any bounded function b satisfying (1.2),

$$|\mathcal{S}_x^b G_D(x, y)| \leq |\mathcal{S}_x^b| G_D(x, y) \leq C_{20} \|b\|_\infty h_D(x, y). \quad (4.34)$$

For every $x, y \in D$ with $x \neq y$, define

$$g_D(x, y) := |x - y|^{\alpha-d} \left(1 \wedge \frac{\delta_D(x)}{|x - y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2}.$$

Lemma 4.7. *Let D be a bounded $C^{1,1}$ open set. There is a constant $C_{21} = C_{21}(d, \alpha, \beta) > 0$ such that for every $x, y, z \in D$,*

(i) *if $\alpha > 2\beta$, then*

$$\frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21} \text{diam}(D)^{\alpha/2-\beta} \left(\frac{1}{|x - z|^{d-\alpha/2}} + \frac{1}{|y - z|^{d-\alpha/2}} \right); \quad (4.35)$$

if $\alpha = 2\beta$, then for every $\theta \in (0, \beta)$,

$$\frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21}(\text{diam}(D)^\theta + \theta^{-1}) \left(\frac{1}{|x - z|^{d-\beta+\theta}} + \frac{1}{|y - z|^{d-\beta+\theta}} \right); \quad (4.36)$$

if $\alpha < 2\beta$, then

$$\frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} \leq C_{21} \left(\frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}} \right). \quad (4.37)$$

(ii) *For every $0 < \alpha < \beta < 2$,*

$$\frac{h_D(x, z)h_D(z, y)}{h_D(x, y)} \leq C_{21} \left(\frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}} \right); \quad (4.38)$$

Proof. (i) Let $f_D(x, y) := |x - y|^{-d+\alpha-\beta} \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}}\right)$ for any $x, y \in D, x \neq y$. Using (4.18), we have

$$1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \asymp \frac{\delta_D(y)^{\alpha/2}}{r_D(x, y)^{\alpha/2}},$$

$$\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x - y|^{\alpha/2}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x - y|^{\alpha/2}}\right) \asymp \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{r_D(x, y)^\alpha},$$

and thus

$$\frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} \asymp \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha}|y - z|^{d-\alpha+\beta}} \left[\frac{\delta_D(z)r_D(x, y)}{r_D(y, z)r_D(x, z)} \right]^{\alpha/2} \left[\frac{r_D(x, y)}{r_D(x, z)} \right]^{\alpha/2}. \quad (4.39)$$

Note that

$$\frac{\delta_D(z)r_D(x, y)}{r_D(y, z)r_D(x, z)} \leq \frac{\delta_D(z)(r_D(x, z) + r_D(y, z))}{r_D(y, z)r_D(x, z)} = \frac{\delta_D(z)}{r_D(y, z)} + \frac{\delta_D(z)}{r_D(x, z)} \leq 2, \quad (4.40)$$

and

$$\frac{r_D(x, y)}{r_D(x, z)} \asymp \frac{|x - y| + \delta_D(x)}{|x - z| + \delta_D(x)} \leq 1 + \frac{|x - y|}{|x - z|}. \quad (4.41)$$

If $|x - z| > |x - y|/2$, then (4.41) ≤ 3 , and consequently by (4.40)

$$\begin{aligned} \frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} &\lesssim \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha}|y - z|^{d-\alpha+\beta}} \\ &\lesssim \frac{|x - z|^{d-\alpha} + |y - z|^{d-\alpha}}{|x - z|^{d-\alpha}|y - z|^{d-\alpha+\beta}} \\ &\lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}} \end{aligned} \quad (4.42)$$

Otherwise if $|x - z| \leq |x - y|/2$, then (4.41) $\leq \frac{3}{2}|x - y|/|x - z|$, and consequently

$$\begin{aligned} \frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} &\lesssim \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha}|y - z|^{d-\alpha+\beta}} \frac{|x - y|^{\alpha/2}}{|x - z|^{\alpha/2}} \\ &= \frac{|x - y|^{d-\alpha/2}}{|x - z|^{d-\alpha/2}|y - z|^{d-\alpha+\beta}} \\ &\lesssim \frac{1}{|y - z|^{d-\alpha+\beta}} + \frac{1}{|x - z|^{d-\alpha/2}|y - z|^{\beta-\alpha/2}}. \end{aligned} \quad (4.43)$$

If $\alpha > 2\beta$, then

$$\begin{aligned} (4.43) &= \frac{|y - z|^{\alpha/2-\beta}}{|y - z|^{d-\alpha/2}} + \frac{|y - z|^{\alpha/2-\beta}}{|x - z|^{d-\alpha/2}} \\ &\leq \text{diam}(D)^{\alpha/2-\beta} \left(\frac{1}{|x - z|^{d-\alpha/2}} + \frac{1}{|y - z|^{d-\alpha/2}} \right). \end{aligned} \quad (4.44)$$

Since $h_D(z, y) = f_D(z, y)$ in this case, (4.35) of Lemma 4.7 comes from (4.43) and (4.44). If $\alpha \leq 2(<?)\beta$, then

$$(4.43) \lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}}. \quad (4.45)$$

and consequently,

$$\frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} \lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}}. \quad (4.46)$$

If $\alpha = 2\beta$, by (4.46) we have

$$\begin{aligned} \frac{g_D(x, z)h_D(z, y)}{g_D(x, y)} &= \frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} \left(1 \vee \log \frac{|y - z|}{\delta_D(z)} \right) \\ &= \frac{g_D(x, z)f_D(z, y)}{g_D(x, y)} \left(1_{\{|y-z| \leq e\delta_D(z)\}} + \log \frac{|y - z|}{\delta_D(z)} 1_{\{|y-z| > e\delta_D(z)\}} \right) \\ &\lesssim \frac{|x - y|^{d-2\beta}}{|x - z|^{d-2\beta}|y - z|^{d-\beta}} \frac{\delta_D(z)^\beta r_D(x, y)^{2\beta}}{r_D(y, z)^\beta r_D(x, z)^{2\beta}} \log \frac{|y - z|}{\delta_D(z)} 1_{\{|y-z| > e\delta_D(z)\}} \\ &\quad + \left(\frac{1}{|x - z|^{d-\beta}} + \frac{1}{|y - z|^{d-\beta}} \right) 1_{\{|y-z| \leq e\delta_D(z)\}} \\ &=: I + II. \end{aligned} \quad (4.47)$$

Fix an arbitrary $\theta \in (0, \beta)$. Note that when $|y - z| > e\delta_D(z)$,

$$\begin{aligned}
\frac{\delta_D(z)^\beta r_D(x, y)^{2\beta}}{r_D(y, z)^\beta r_D(x, z)^{2\beta}} \log \frac{|y - z|}{\delta_D(z)} &\lesssim \frac{\delta_D(z)^\beta}{r_D(y, z)^\beta} \log \frac{|y - z|}{\delta_D(z)} + \frac{\delta_D(z)^\beta r_D(y, z)^\beta}{r_D(x, z)^{2\beta}} \log \frac{|y - z|}{\delta_D(z)} \\
&\lesssim 1 + \frac{\delta_D(z)^{\beta-\theta}}{r_D(x, z)^{\beta-\theta}} \frac{|y - z|^{\beta+\theta}}{|x - z|^{\beta+\theta}} \left(\frac{\delta_D(z)}{|y - z|} \right)^\theta \log \frac{|y - z|}{\delta_D(z)} \\
&\leq 1 + \theta^{-1} \frac{|y - z|^{\beta+\theta}}{|x - z|^{\beta+\theta}}, \tag{4.48}
\end{aligned}$$

The last inequality comes from the fact that $g(x) := (x^{-\theta} \log x) 1_{\{x > e\}}$ is bounded from above by θ^{-1} . Consequently

$$\begin{aligned}
I &\lesssim \frac{|x - y|^{d-2\beta}}{|x - z|^{d-2\beta} |y - z|^{d-\beta}} \left(1 + \theta^{-1} \frac{|y - z|^{\beta+\theta}}{|x - z|^{\beta+\theta}} \right) 1_{\{|y - z| > e\delta_D(z)\}} \\
&\lesssim 1_{\{|y - z| > e\delta_D(z)\}} \left(\frac{1}{|x - z|^{d-\beta}} + \frac{1}{|y - z|^{d-\beta}} + \frac{\theta^{-1}}{|x - z|^{d-\beta+\theta}} + \frac{\theta^{-1}}{|y - z|^{d-\beta+\theta}} \right). \tag{4.49}
\end{aligned}$$

Thus by (4.47) and (4.49) we have

$$\frac{g_D(x, z) h_D(z, y)}{g_D(x, y)} \lesssim (\text{diam}(D)^\theta + \theta^{-1}) \left(\frac{1}{|x - z|^{d-\beta+\theta}} + \frac{1}{|y - z|^{d-\beta+\theta}} \right).$$

So we get (4.36) of Lemma 4.7. If $\alpha < 2\beta$, note that

$$\begin{aligned}
&\frac{g_D(x, z) h_D(z, y)}{g_D(x, y)} \\
&= \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} \left(1 \vee \frac{|y - z|^{\beta-\alpha/2}}{\delta_D(z)^{\beta-\alpha/2}} \right) \\
&= \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} 1_{\{\delta_D(z) \geq |y - z|\}} + \frac{g_D(x, z) f_D(z, y)}{g_D(x, y)} \frac{|y - z|^{\beta-\alpha/2}}{\delta_D(z)^{\beta-\alpha/2}} 1_{\{\delta_D(z) < |y - z|\}}. \\
&=: III + IV. \tag{4.50}
\end{aligned}$$

obviously (4.46) implies that

$$III \lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}}. \tag{4.51}$$

For IV , since $r_D(y, z) \asymp |y - z|$ for $y, z \in D$ with $\delta_D(z) < |y - z|$, we have

$$\begin{aligned}
IV &\asymp \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha} |y - z|^{d-\alpha+\beta}} \frac{\delta_D(z)^{\alpha/2} r_D(x, y)^\alpha}{r_D(y, z)^{\alpha/2} r_D(x, z)^\alpha} \frac{|y - z|^{\beta-\alpha/2}}{\delta_D(z)^{\beta-\alpha/2}} 1_{\{\delta_D(z) < |y - z|\}} \\
&\asymp \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha} |y - z|^{d-\alpha+\beta}} \frac{\delta_D(z)^{\alpha-\beta} r_D(x, y)^\alpha}{|y - z|^{\alpha-\beta} r_D(x, z)^\alpha} 1_{\{\delta_D(z) < |y - z|\}}, \tag{4.52}
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{\delta_D(z)^{\alpha-\beta} r_D(x, y)^\alpha}{|y-z|^{\alpha-\beta} r_D(x, z)^\alpha} 1_{\{\delta_D(z) < |y-z|\}} \\
& \lesssim \frac{\delta_D(z)^{\alpha-\beta}}{|y-z|^{\alpha-\beta}} \left(1 + \frac{r_D(y, z)^\alpha}{r_D(x, z)^\alpha} \right) 1_{\{\delta_D(z) < |y-z|\}} \\
& \leq 1 + \frac{\delta_D(z)^{\alpha-\beta}}{r_D(x, z)^{\alpha-\beta}} \frac{r_D(y, z)^\alpha}{|y-z|^{\alpha-\beta} r_D(x, z)^\beta} 1_{\{\delta_D(z) < |y-z|\}} \\
& \leq 1 + \frac{|y-z|^\beta}{|x-z|^\beta}.
\end{aligned} \tag{4.53}$$

Thus

$$\begin{aligned}
(4.52) & \lesssim \frac{|x-y|^{d-\alpha}}{|x-z|^{d-\alpha} |y-z|^{d-\alpha+\beta}} + \frac{|x-y|^{d-\alpha}}{|x-z|^{d-\alpha+\beta} |y-z|^{d-\alpha}} \\
& \lesssim \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}}.
\end{aligned} \tag{4.54}$$

By (4.50), (4.51), (4.52) and (4.54) we proved (4.37) for $\alpha < 2\beta$.

(ii) If $\alpha > 2\beta$, we have

$$\begin{aligned}
\frac{h_D(x, z) h_D(z, y)}{h_D(x, y)} & \asymp \frac{|x-y|^{d-\alpha+\beta}}{|x-z|^{d-\alpha+\beta} |y-z|^{d-\alpha+\beta}} \left(\frac{\delta_D(z) r_D(x, y)}{r_D(x, z) r_D(y, z)} \right)^{\alpha/2} \\
& \lesssim \frac{|x-y|^{d-\alpha+\beta}}{|x-z|^{d-\alpha+\beta} |y-z|^{d-\alpha+\beta}} \\
& \lesssim \frac{1}{|x-z|^{d-\alpha+\beta}} + \frac{1}{|y-z|^{d-\alpha+\beta}}.
\end{aligned} \tag{4.55}$$

If $\alpha = 2\beta$, we have

$$\begin{aligned}
& \frac{h_D(x, z) h_D(z, y)}{h_D(x, y)} \\
& \asymp \frac{|x-y|^{d-\beta}}{|x-z|^{d-\beta} |y-z|^{d-\beta}} \frac{r_D(x, y)^\beta \delta_D(z)^\beta}{r_D(x, z)^\beta r_D(y, z)^\beta} \frac{\left(1 \vee \log \frac{|x-z|}{\delta_D(x)} \right) \left(1 \vee \log \frac{|y-z|}{\delta_D(z)} \right)}{\left(1 \vee \log \frac{|y-x|}{\delta_D(x)} \right)} \\
& = \frac{|x-y|^{d-\beta}}{|x-z|^{d-\beta} |y-z|^{d-\beta}} \frac{r_D(x, y)^\beta \delta_D(z)^\beta}{r_D(x, z)^\beta r_D(y, z)^\beta} \left(\log \frac{|x-z|}{\delta_D(x)} 1_{\{|x-z| \geq e\delta_D(x), |y-z| < e\delta_D(z), |y-x| < e\delta_D(x)\}} \right. \\
& \quad + 1_{\{|x-z| < e\delta_D(x), |y-z| < e\delta_D(z), |y-x| < e\delta_D(x)\}} + \log \frac{|y-z|}{\delta_D(z)} 1_{\{|x-z| < e\delta_D(x), |y-z| \geq e\delta_D(z), |y-x| < e\delta_D(x)\}} \\
& \quad \left. + \log \frac{|x-z|}{\delta_D(x)} \log \frac{|y-z|}{\delta_D(z)} 1_{\{|x-z| \geq e\delta_D(x), |y-z| \geq e\delta_D(z), |y-x| < e\delta_D(x)\}} \right).
\end{aligned} \tag{4.56}$$

First we note that

$$\frac{r_D(x, y)^\beta \delta_D(z)^\beta}{r_D(x, z)^\beta r_D(y, z)^\beta} 1_{\{|x-z| < e\delta_D(x), |y-z| < e\delta_D(z), |y-x| < e\delta_D(x)\}} \lesssim \frac{\delta_D(z)^\beta}{r_D(x, z)^\beta} + \frac{\delta_D(z)^\beta}{r_D(y, z)^\beta} \leq 2.$$

Since $f(x) = (x^{-\beta} \log x)1_{\{x \geq e\}}$ is bounded from above, we have

$$\begin{aligned} & \frac{r_D(x, y)^\beta \delta_D(z)^\beta}{r_D(x, z)^\beta r_D(y, z)^\beta} \log \frac{|x - z|}{\delta_D(x)} 1_{\{|x - z| \geq e\delta_D(x), |y - z| < e\delta_D(z), |y - x| < e\delta_D(x)\}} \\ & \lesssim \frac{\delta_D(z)^\beta}{r_D(y, z)^\beta} \left(\frac{\delta_D(x)}{|x - z|} \right)^\beta \log \frac{|x - z|}{\delta_D(x)} 1_{\{|x - z| \geq e\delta_D(x), |y - z| < e\delta_D(z), |y - x| < e\delta_D(x)\}} \\ & \lesssim 1. \end{aligned}$$

Applying similar calculations to the remaining two terms in the bracket of (4.56), we get

$$\begin{aligned} \frac{h_D(x, z)h_D(z, y)}{h_D(x, y)} & \lesssim \frac{|x - y|^{d-\beta}}{|x - z|^{d-\beta}|y - z|^{d-\beta}} \\ & \lesssim \frac{1}{|x - z|^{d-\beta}} + \frac{1}{|y - z|^{d-\beta}}. \end{aligned} \quad (4.57)$$

If $\alpha < 2\beta$, by (4.18) we have

$$1 \vee \frac{|x - y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp 1 + \frac{|x - y|^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}} \asymp \frac{r_D(x, y)^{\beta-\alpha/2}}{\delta_D(x)^{\beta-\alpha/2}},$$

and thus

$$h_D(x, y) \asymp |x - y|^{-d+\alpha-\beta} \frac{\delta_D(y)^{\alpha/2}}{r_D(x, y)^{\alpha-\beta} \delta_D(x)^{\beta-\alpha/2}}.$$

It follows from (4.40) that

$$\begin{aligned} \frac{h_D(x, z)h_D(z, y)}{h_D(x, y)} & \asymp \frac{|x - y|^{d-\alpha+\beta}}{|x - z|^{d-\alpha+\beta}|y - z|^{d-\alpha+\beta}} \left(\frac{\delta_D(z)r_D(x, y)}{r_D(x, z)r_D(y, z)} \right)^{\alpha-\beta} \\ & \lesssim \frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}}. \end{aligned} \quad (4.58)$$

Therefore we complete the proof of (4.38). \square

Definition 4.8. Suppose $\gamma > 0$. For a function f defined on \mathbb{R}^d , we define for $r > 0$,

$$M_f^\gamma(r) = \sup_{x \in \mathbb{R}^d} \int_{B(x, r)} \frac{|f(y)|}{|x - y|^{d-\gamma}} dy.$$

f is said to belong to the Kato class $\mathbb{K}_{d, \gamma}$ if $\lim_{r \downarrow 0} M_f^\gamma(r) = 0$. For any bounded open set $D \subset \mathbb{R}^d$, we define

$$M_f^\gamma(D) := \sup_{x \in \mathbb{R}^d} \int_D \frac{|f(y)|}{|x - y|^{d-\gamma}} dy.$$

Lemma 4.9. Let D be a bounded $C^{1,1}$ open set. Then for any bounded function b satisfying (1.2),

$$\mathcal{S}_x^b \int_D G_D(x, z) \mathcal{S}_z^b G_D(z, y) dz = \int_D \mathcal{S}_x^b G_D(x, z) \mathcal{S}_z^b G_D(z, y) dz, \quad \forall x, y \in D, x \neq y. \quad (4.59)$$

Furthermore, let $\gamma := (\alpha - \beta) \wedge (\alpha/2)$ if $\alpha/2 \neq \beta$ and $\gamma \in (0, \beta)$ if $\alpha/2 = \beta$. Then for any measurable function $f \in \mathbb{K}_{d, \gamma}$,

$$\mathcal{S}_x^b \int_D G_D(x, z) f(z) dz = \int_D \mathcal{S}_x^b G_D(x, z) f(z) dz, \quad \forall x \in D. \quad (4.60)$$

Proof. Fix $x, y \in D, x \neq y$. For any $\varepsilon > 0$,

$$\begin{aligned}
& \left| (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} \mathcal{S}_z^b G_D(z, y) \right| 1_{\{z \in D, |u| > \varepsilon\}} \\
& \leq \|b\|_\infty |u|^{-d-\beta} |\mathcal{S}_z^b G_D(z, y)| (G_D(x+u, z) 1_{\{z, x+u \in D, |u| > \varepsilon\}} + G_D(x, z) 1_{\{z \in D, |u| > \varepsilon\}}) \\
& \leq C_{16} C_{20} \|b\|_\infty |u|^{-d-\beta} h_D(z, y) \left(|x+u-z|^{\alpha-d} 1_{\{z, x+u \in D, |u| > \varepsilon\}} + |x-z|^{\alpha-d} 1_{\{z \in D, |u| > \varepsilon\}} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_{|u| > \varepsilon} \int_{z \in D} \left| (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} \mathcal{S}_z^b G_D(z, y) \right| dz du \\
& \leq C_{16} C_{20} \|b\|_\infty \left(\int_{\substack{v \in D \\ |v-x| > \varepsilon}} \int_{z \in D} |v-x|^{-d-\beta} |v-z|^{-d+\alpha} h_D(z, y) dz dv \right. \\
& \quad \left. + \int_{|u| > \varepsilon} \int_{z \in D} |u|^{-d-\beta} |z-x|^{-d+\alpha} h_D(z, y) dz du \right). \tag{4.61}
\end{aligned}$$

It is not hard to prove the integrals in (4.61) are finite. Thus the integral

$$\int_{|u| > \varepsilon} \int_{z \in D} (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} \mathcal{S}_z^b G_D(z, y) dz du$$

is absolutely convergent. By Fubini's theorem, we have

$$\begin{aligned}
& \mathcal{S}_x^b \int_D G_D(x, z) \mathcal{S}_z^b G_D(z, y) dz \\
& = \lim_{\varepsilon \rightarrow 0} \int_{|u| > \varepsilon} \int_{z \in D} (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} \mathcal{S}_z^b G_D(z, y) dz du \\
& = \lim_{\varepsilon \rightarrow 0} \int_{z \in D} \left[\int_{|u| > \varepsilon} (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} du \right] \mathcal{S}_z^b G_D(z, y) dz \\
& = \int_{z \in D} \lim_{\varepsilon \rightarrow 0} \left[\int_{|u| > \varepsilon} (G_D(x+u, z) - G_D(x, z))b(x, u)|u|^{-d-\beta} du \right] \mathcal{S}_z^b G_D(z, y) dz \\
& = \int_D \mathcal{S}_x^b G_D(x, z) \mathcal{S}_z^b G_D(z, y) dz.
\end{aligned}$$

Here the third equality follows from dominated convergence theorem since for $\lambda = (\delta_D(x) \wedge |x - z|)/2$ and $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
& \left| \int_{|u| > \varepsilon} (G_D(x+u, z) - G_D(x, z)) b(x, u) |u|^{-d-\beta} du \right| \\
& = \left| \int_{|u| > \varepsilon} (G_D(x+u, z) - G_D(x, z) - \nabla_x G_D(x, z) \cdot u 1_{|u| < \lambda}) b(x, u) |u|^{-d-\beta} du \right| \\
& \leq \int_{|u| < \lambda} |G_D(x+u, z) - G_D(x, z) - \nabla_x G_D(x, z) \cdot u| |b(x, u)| |u|^{-d-\beta} du \\
& \quad + \int_{|u| \geq \lambda} (G_D(x+u, z) + G_D(x, z)) |b(x, u)| |u|^{-d-\beta} du \\
& = |\mathcal{S}_x^b G_D(x, z)|,
\end{aligned}$$

and by (4.34) and (4.38)

$$\begin{aligned}
& \int_{z \in D} |\mathcal{S}_x^b| G_D(x, z) |\mathcal{S}_z^b| G_D(z, y) dz \\
& \leq C_{20}^2 \int_{z \in D} h_D(x, z) h_D(z, y) dz \\
& \leq C_{20}^2 C_{21} h_D(x, y) \int_{z \in D} |x - z|^{-d+\alpha-\beta} + |z - y|^{-d+\alpha-\beta} dz < \infty.
\end{aligned}$$

Hence we get (4.59). To prove (4.60), first we note that $\mathbb{K}_{d,\gamma} \subset \mathbb{K}_{d,\alpha-\beta} \subset \mathbb{K}_{d,\alpha}$. For any $\varepsilon > 0$ and $x \in D$, we have

$$\begin{aligned}
& \int_{|u|>\varepsilon} \int_D |G_D(x+u, z) - G_D(x, z)| \frac{|b(x, z)|}{|u|^{d+\beta}} |f(z)| dz du \\
& \leq \|b\|_\infty \left(\int_{|u|>\varepsilon, u+x \in D} \int_D G_D(x+u, z) |u|^{-d-\beta} |f(z)| dz du \right. \\
& \quad \left. + \int_{|u|>\varepsilon} \int_D G_D(x, z) |u|^{-d-\beta} |f(z)| dz du \right) \\
& \leq \|b\|_\infty C_{16} \left[\varepsilon^{-d-\beta} \int_{|u|>\varepsilon, u+x \in D} \left(\int_D |x+u-z|^{\alpha-d} |f(z)| dz \right) du \right. \\
& \quad \left. + \int_{|u|>\varepsilon} |u|^{-d-\beta} \left(\int_D |f(z)| |x-z|^{\alpha-d} dz \right) du \right] \\
& \lesssim \|b\|_\infty C_{16} \left(\varepsilon^{-d-\beta} (\text{diam}(D) + |x|)^d + \varepsilon^{-\beta} \right) M_f^\alpha(D) < \infty.
\end{aligned}$$

In other words, $\int_{|u|>\varepsilon} \int_D (G_D(x+u, z) - G_D(x, z)) \frac{b(x, z)}{|u|^{d+\beta}} f(z) dz du$ is absolutely convergent. We observe that for any $\gamma > 0$ satisfying our assumption, $h_D(x, y) \leq c_1 |x - y|^{-d+\gamma}$ for some positive constant $c_1 = c_1(d, \alpha, \beta, \text{diam}(D), \delta_D(x))$, thus

$$\int_D h_D(x, y) |f(y)| dy \leq c_1 M_f^\gamma(D) < \infty.$$

Therefore, we can apply similar arguments as in the proof of (4.59) to get (4.60). \square

Lemma 4.10. *Let D be a bounded $C^{1,1}$ open set in \mathbb{R}^d . Then for any bounded function b satisfying (1.2) and (1.4), we have*

$$G_D^b(x, y) = G_D(x, y) + \int_D G_D^b(x, z) \mathcal{S}_z^b G_D(z, y) dz, \quad \forall x, y \in D. \quad (4.62)$$

Proof. In view of Lemma 4.2 and Lemma 4.6, it is easy to show that the integral on the right hand side of (4.62) is absolutely convergent, and is continuous in $y \in D \setminus \{x\}$ for every $x \in D$. The analogous formula of [2, (41)] also holds for the operator \mathcal{L}^b , that is, for every $\phi \in C_c^\infty(D)$ and $x \in D$,

$$\int_D G_D^b(x, z) \mathcal{L}_z^b \phi(z) dz = -\phi(x). \quad (4.63)$$

With (4.60) and (4.63), we can repeat the arguments in [2, Lemma 12] (with $b(z)\nabla$ replaced by \mathcal{S}_z^b and \tilde{G} by G_D^b) to get Lemma 4.10. \square

Theorem 4.11. *Suppose $A \in (0, \infty)$. There exists a positive constant $r_1 = r_1(d, \alpha, \beta, A)$ and $C_{22} = C_{22}(d, \alpha, \beta, A)$ such that for any bounded function b satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$, any ball $B = B(x_0, r)$ with radius $0 < r \leq r_1$,*

$$\frac{1}{2}G_B(x, y) \leq G_B^b(x, y) \leq \frac{3}{2}G_B(x, y) \quad \text{and} \quad |\mathcal{S}_x^b G_B^b(x, y)| \leq C_{22}h_B(x, y) \quad (4.64)$$

for $x, y \in B$. Moreover, we have

$$G_B^b(x, y) = \sum_{k=0}^{\infty} G_k(x, y), \quad (4.65)$$

where

$$G_0(x, y) := G_B(x, y) \quad \text{and} \quad G_n(x, y) := \int_B G_{n-1}(x, z) \mathcal{S}_z^b G_B(z, y) dz \quad \text{for } n \geq 1, \quad (4.66)$$

and the constant r_1 satisfies the following property:

$$r_1(d, \alpha, \beta, A_\lambda) = \lambda r_1(d, \alpha, \beta, A), \quad \forall \lambda > 0. \quad (4.67)$$

Proof. (4.67) follows directly from (4.64) and the scaling property for G_B^b . We only need to show (4.64). Without loss of generality, we may assume $r_1 \in (0, 1]$. By (4.8) and (4.34), one can find positive constants $c_1 = c_1(d, \alpha)$ and $c_2 = c_2(d, \alpha, \beta)$ such that for any ball B with radius r and $x, y \in B$

$$G_B(x, y) \stackrel{c_1}{\asymp} g_B(x, y), \quad (4.68)$$

and

$$|\mathcal{S}_x^b G_B(x, y)| \leq c_2 A h_B(x, y). \quad (4.69)$$

Let $\gamma := (\alpha - \beta) \wedge \alpha/2$ for $\alpha/2 \neq \beta$ and $\gamma := \beta/2$ for $\alpha/2 = \beta$. Note that by Lemma 4.7, we have

$$\begin{aligned} & \int_B g_B(x, z) h_B(z, y) dz \\ & \leq 3C_{21} g_B(x, y) \int_B \left(\frac{1}{|x - z|^{d-\gamma}} + \frac{1}{|y - z|^{d-\gamma}} \right) dz \\ & \leq 6C_{21} \gamma^{-1} r^\gamma g_B(x, y) \\ & =: C(r) g_B(x, y), \end{aligned} \quad (4.70)$$

and similarly,

$$\begin{aligned} & \int_B h_B(x, z) h_B(z, y) dz \\ & \leq C_{21} h_B(x, y) \int_B \left(\frac{1}{|x - z|^{d-\alpha+\beta}} + \frac{1}{|y - z|^{d-\alpha+\beta}} \right) dz \\ & \leq 2C_{21} (\alpha - \beta)^{-1} r^{\alpha-\beta} h_B(x, y) \\ & \leq C(r) h_B(x, y). \end{aligned} \quad (4.71)$$

Let $G_k(x, y)$ be defined by (4.66). By the above, (4.59), (4.68), and (4.69), we have for all $x, y \in B$

$$\begin{aligned} |G_1(x, y)| &\leq \int_B G_B(x, z) |\mathcal{S}_z^b G_B(z, y)| dz \\ &\leq c_1 c_2 A \int_B g_B(x, z) h_B(z, y) dz \\ &\leq c_1 c_2 AC(r) g_B(x, y), \end{aligned} \quad (4.72)$$

and

$$\begin{aligned} |\mathcal{S}_x^b G_1(x, y)| &\leq \int_B \left| \mathcal{S}_x^b G_B(x, z) \mathcal{S}_z^b G_B(z, y) \right| dz \\ &\leq (c_2 A)^2 \int_B h_B(x, z) h_B(z, y) dz \\ &\leq (c_2 A)^2 C(r) h_B(x, y). \end{aligned} \quad (4.73)$$

Note that for every $n \geq 1$, we have

$$G_n(x, y) = \int_B G_B(x, z) \mathcal{S}_z^b G_{n-1}(z, y) dz, \quad (4.74)$$

and

$$\mathcal{S}_x^b G_n(x, y) = \int_B \mathcal{S}_x^b G_{n-1}(x, z) \mathcal{S}_z^b G_B(z, y) dz. \quad (4.75)$$

The above equalities are proved consecutively by induction. Thus by (4.70), (4.71) and induction, we have

$$|G_n(x, y)| \leq c_1 (c_2 AC(r))^n g_B(x, y) \leq c_1^2 (c_2 AC(r))^n G_B(x, y), \quad (4.76)$$

and

$$|\mathcal{S}_x^b G_n(x, y)| \leq c_2 A (c_2 AC(r))^n h_B(x, y). \quad (4.77)$$

Applying Duhamel's formula (4.62) recursively n times, we get for $n \geq 0$ and $x, y \in B, x \neq y$,

$$G_B^b(x, y) = \sum_{k=0}^n G_k(x, y) + \int_B G_B^b(x, z) \mathcal{S}_z^b G_n(z, y) dz. \quad (4.78)$$

Note that $C(r) = 6C_{21}\gamma^{-1}r^\gamma \downarrow 0$ as $r \downarrow 0$. Now we let $r_1 \in (0, 1]$ be sufficiently small so that $\delta := c_2 AC(r_1) \leq 1/(2c_1^2 + 1)$. By Lemma 4.2 and (4.77), we have for any $r \in (0, r_1]$,

$$\lim_{n \rightarrow \infty} \left| \int_B G_B^b(x, z) \mathcal{S}_z^b G_n(z, y) dz \right| \leq \lim_{n \rightarrow \infty} c_2 A \delta^n C_{16} \int_B |x - z|^{-d+\alpha} h_B(z, y) dz = 0.$$

This together with (4.78) establishes (4.65). The first assertion in (4.64) then follows from the fact that

$$\sum_{n=1}^{\infty} |G_n(x, y)| \leq \sum_{n=1}^{\infty} c_1^2 \delta^n G_B(x, y) \leq G_B(x, y)/2$$

for any $B = B(x_0, r)$ with $r \in (0, r_1]$. We next prove the second assertion in (4.64). Note that by (4.65),

$$\begin{aligned}\mathcal{S}_x^b G_{B_r}^b(x, y) &= \mathcal{A}(d, -\beta) \lim_{\varepsilon \rightarrow 0} \int_{|z| > \varepsilon} \frac{G_{B_r}^b(x + z, y) - G_{B_r}^b(x, y)}{|z|^{d+\beta}} b(x, z) dz \\ &= \mathcal{A}(d, -\beta) \lim_{\varepsilon \rightarrow 0} \int_{|z| > \varepsilon} \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{G_k(x + z, y) - G_k(x, y)}{|z|^{d+\beta}} \right) b(x, z) dz.\end{aligned}\quad (4.79)$$

Note that by (4.77), for any $n \geq 1$,

$$\begin{aligned}\sum_{k=0}^n |G_k(x + z, y) - G_k(x, y)| &\leq \sum_{k=0}^n c_1 \delta^k (G_{B_r}(x + z, y) + G_{B_r}(x, y)) \\ &\leq c_1 c_3 (1 - \delta)^{-1} (|x + z - y|^{\alpha-d} + |x - y|^{\alpha-d}).\end{aligned}$$

The last term is absolutely convergent with respect to $|b(x, z)| |z|^{-d-\beta} dz$ on the domain $\{z \in \mathbb{R}^d : |z| > \varepsilon\}$ for any $\varepsilon > 0$. Thus using the dominated convergence theorem, we can continue the calculation in (4.79) to get

$$\begin{aligned}\mathcal{S}_x^b G_{B_r}^b(x, y) &= \mathcal{A}(d, -\beta) \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{|z| > \varepsilon} \left(\sum_{k=0}^n \frac{G_k(x + z, y) - G_k(x, y)}{|z|^{d+\beta}} \right) b(x, z) dz \\ &= \mathcal{A}(d, -\beta) \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} F_n(\varepsilon),\end{aligned}\quad (4.80)$$

where

$$F_n(\varepsilon) := \sum_{k=0}^n \int_{|z| > \varepsilon} (G_k(x + z, y) - G_k(x, y)) b(x, z) |z|^{-d-\beta} dz$$

for $\varepsilon > 0$. It follows from (4.75), (4.77), Lemma 4.6 and Lemma 4.9 that for any $n, m \in \mathbb{Z}_+$, $n >$

m and any $\lambda, \varepsilon > 0$,

$$\begin{aligned}
& |F_n(\varepsilon) - F_m(\varepsilon)| \\
&= \left| \sum_{k=m+1}^n \int_{|z|>\varepsilon} \frac{G_k(x+z, y) - G_k(x, y)}{|z|^{d+\beta}} b(x, z) dz \right| \\
&= \left| \sum_{k=m+1}^n \int_{|z|>\varepsilon} \int_{B_r} \frac{G_{B_r}(x+z, u) - G_{B_r}(x, u)}{|z|^{d+\beta}} \mathcal{S}_u^b G_{k-1}(u, y) b(x, z) du dz \right| \\
&= \left| \sum_{k=m+1}^n \int_{B_r} \left[\int_{|z|>\varepsilon} \frac{G_{B_r}(x+z, u) - G_{B_r}(x, u) - \nabla G_{B_r}(x, z) \cdot z 1_{\{|z|<\lambda\}}}{|z|^{d+\beta}} b(x, z) dz \right] \mathcal{S}_u^b G_{k-1}(u, y) du \right| \\
&\leq \sum_{k=m+1}^n c_2 A \delta^{k-1} \int_{B_r} \left(\int_{|z|<\lambda} \frac{|G_{B_r}(x+z, u) - G_{B_r}(x, u) - \nabla G_{B_r}(x, z) \cdot z|}{|z|^{d+\beta}} |b(x, z)| dz \right. \\
&\quad \left. + \int_{|z|\geq\lambda} \frac{G_{B_r}(x+z, u) + G_{B_r}(x, u)}{|z|^{d+\beta}} |b(x, z)| dz \right) h_{B_r}(u, y) du \\
&\leq c_2 c_4 \sum_{k=m+1}^n \delta^{k-1} \int_{B_r} h_{B_r}(x, u) h_{B_r}(u, y) du \\
&\leq c_2 c_4 c_5 h_{B_r}(x, y) \sum_{k=m+1}^n \delta^{k-1} \int_{B_r} (|x-u|^{-d+\alpha-\beta} + |y-u|^{-d+\alpha-\beta}) du \\
&\leq c_6 h_{B_r}(x, y) \sum_{k=m+1}^n \delta^{k-1}.
\end{aligned}$$

Here $c_i = c_i(d, \alpha, \beta, A) > 0$, $i = 4, 5, 6$. Therefore $\sup_{\varepsilon>0} |F_n(\varepsilon) - F_m(\varepsilon)| \rightarrow 0$ as $m, n \rightarrow \infty$. This implies that $\{F_n(\varepsilon) : n \geq 1\}$ is an uniformly convergent sequence of continuous functions. It follows that

$$\mathcal{S}_x^b G_{B_r}^b(x, z) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} F_n(\varepsilon) = \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} F_n(\varepsilon) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \mathcal{S}_x^b G_k(x, z) = \sum_{k=0}^{\infty} \mathcal{S}_x^b G_k(x, z).$$

The second assertion in (4.64) now follows from estimate (4.77). \square

The proof for the following lemma is similar to that for the first assertion in (4.64). We omit the details here.

Lemma 4.12. *Suppose D is a bounded $C^{1,1}$ open set in \mathbb{R}^d with characteristic (R_0, Λ_0) and $A \in (0, \infty)$. There exists a positive constant $r_2 = r_2(d, \alpha, \beta, D, A) \in (0, R_0)$ such that for every bounded function b satisfying (1.2) and (1.4) with $\|b\|_{\infty} \leq A$, every $Q \in \partial D$ and $r \in (0, r_2]$, we have*

$$\frac{1}{2} G_{V(Q,r)}(x, y) \leq G_{V(Q,r)}^b(x, y) \leq \frac{3}{2} G_{V(Q,r)}(x, y), \quad \forall x, y \in V(Q, r).$$

Moreover r_2 satisfies that $r_2(d, \alpha, \beta, \lambda D, A_{\lambda}) = \lambda r_2(d, \alpha, \beta, D, A)$ for any $\lambda > 0$.

It follows from (3.1) that for every bounded open set D in \mathbb{R}^d , every $f \geq 0$, and $x \in D$,

$$\mathbb{E}_x \left(f(X_{\tau_D^b}^b) : X_{\tau_D^b-}^b \neq X_{\tau_D^b}^b \right) = \int_{\bar{D}^c} f(z) \left(\int_D G_D^b(x, y) j^b(y, z) dy \right) dz. \quad (4.81)$$

Define

$$K_D^b(x, z) := \int_D G_D^b(x, y) j^b(y, z) dy, \quad \forall (x, z) \in D \times \bar{D}^c. \quad (4.82)$$

Then (4.81) can be rewritten as

$$\mathbb{E}_x \left(f(X_{\tau_D^b}^b) : X_{\tau_D^b-}^b \neq X_{\tau_D^b}^b \right) = \int_{\bar{D}^c} f(z) K_D^b(x, z) dz. \quad (4.83)$$

Lemma 4.13. *Suppose D is a $C^{1,1}$ open set with $\text{diam}(D) \leq r_3 := \frac{1}{4}\varepsilon(A) \wedge 3r_1$. Here r_1 is the constant defined in Theorem 4.11. Then*

$$\mathbb{P}_x \left(X_{\tau_D^b}^b \in \partial D \right) = 0, \quad \forall x \in D.$$

In this case, for every non-negative measurable function f ,

$$\mathbb{E}_x f(X_{\tau_D^b}^b) = \int_{\bar{D}^c} f(z) K_D^b(x, z) dz \quad \forall x \in D.$$

Proof. Fix $x \in D$. Set $r = \frac{1}{2}(\delta_D(x) \wedge r_1)$. Obviously $\frac{1}{2}\delta_D(x) \geq r \geq \frac{1}{2}(\delta_D(x) \wedge \frac{1}{3}\text{diam}(D)) \geq \frac{1}{12}\delta_D(x)$. Let $B := B(x, r) \subset D$. Since $\text{diam}(D) \leq \varepsilon(A)/4$, by the inner and outer cone property of Lipschitz domains we can find a ball $B' = B(x_0, r) \subset \{z \in \bar{D}^c : \text{dist}(z, D) < \varepsilon(A)/2\}$ such that its distance to B is comparable with r , i.e. for every $y \in B$ and $z \in B'$, $|y - z| \asymp r$. Note that for every $y \in B$ and $z \in B'$, $|y - z| < \varepsilon(A)$. It follows from (4.83), Theorem 4.11, (3.4) and (4.7) that

$$\begin{aligned} \mathbb{P}_x \left(X_{\tau_B^b}^b \in B' \right) &= \int_{B'} \int_B G_B^b(x, y) j^b(y, z) dy dz \\ &\geq \frac{1}{4} \int_{B'} \int_B G_B(x, y) j(y, z) dy dz \\ &= \frac{1}{4} \int_{B'} K_B(x, z) dz \\ &\geq c > 0 \end{aligned} \quad (4.84)$$

for some constant $c = c(d, \alpha) > 0$. Let $D_n := \{y \in D : \delta_D(y) > 1/n\}$ for every $n \in \mathbb{N}$. For n sufficiently large, we have $B \subset D_n$. In this case

$$\begin{aligned} \mathbb{P}_x \left(X_{\tau_{D_n}^b}^b \in \bar{D} \right) &= \mathbb{P}_x \left(X_{\tau_B^b}^b \in \bar{D} \setminus D_n \right) + \mathbb{P}_x \left(X_{\tau_B^b}^b \in D_n \setminus B, X_{\tau_{D_n}^b}^b \in \bar{D} \right) \\ &\leq \mathbb{P}_x \left(X_{\tau_B^b}^b \in \bar{D} \setminus B \right) \\ &\leq 1 - \mathbb{P}_x \left(X_{\tau_B^b}^b \in B' \right) \leq 1 - c. \end{aligned}$$

Let $u(x) = \mathbb{P}_x \left(X_{\tau_D^b}^b \in \partial D \right)$ and $C := \sup\{u(x) : x \in D\}$. By the strong Markov property,

$$\begin{aligned} u(x) &= \mathbb{P}_x \left(u(X_{\tau_{D_n}^b}^b) : X_{\tau_{D_n}^b}^b \in \bar{D} \right) \\ &= \mathbb{P}_x \left(u(X_{\tau_{D_n}^b}^b) : X_{\tau_{D_n}^b}^b \in \partial D \right) + \mathbb{P}_x \left(u(X_{\tau_{D_n}^b}^b) : X_{\tau_{D_n}^b}^b \in D \right). \end{aligned}$$

Since $\mathbb{P}_x \left(X_{\tau_{D_n}^b}^b \in \partial D \right) = 0$ by (3.1), we get

$$u(x) = \mathbb{P}_x \left(u(X_{\tau_{D_n}^b}^b) : X_{\tau_{D_n}^b}^b \in D \right),$$

and consequently $C \leq (1 - c)C$. Thus $C = 0$. \square

5 Duality

In this section, we assume that E is an arbitrary open ball in \mathbb{R}^d . We will discuss some basic properties of $X^{b,E}$ and its dual process under a certain reference measure. By Theorem 3.2 and Lemma 3.5, $X^{b,E}$ has a jointly continuous strictly positive transition density $p_E^b(t, x, y)$. Using the continuity of $p_E^b(t, x, y)$ and the estimates

$$p_E^b(t, x, y) \leq p^b(t, x, y) \leq c_1 e^{c_2 t} \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right),$$

we can easily prove that $X^{b,E}$ is a Hunt process with strong Feller property, *i.e.*, $P_t^{b,E} f(x) := \mathbb{E}_x[f(X_t^{b,E})] \in C_b(E)$ for every $f \in \mathcal{B}_b(E)$.

Define

$$h_E(x) := \int_E G_E^b(y, x) dy, \quad \xi_E(dx) := h_E(x) dx.$$

Proposition 5.1. $h_E(x)$ is a strictly positive, bounded continuous function on E . $\xi_E(dx)$ is an excessive measure for $X^{b,E}$, that is, for any non-negative Borel function f ,

$$\int_E P_t^{b,E} f(x) \xi_E(dx) \leq \int_E f(x) \xi_E(dx).$$

Proof. The first claim follows from (4.11), (4.12) and the continuity and strict positivity of $p_E^b(t, x, y)$. We only need to show the second claim. By Fubini's theorem and Markov property

we have

$$\begin{aligned}
\int_E P_t^{b,E} f(x) \xi_E(dx) &= \int_E \int_E P_t^{b,E} f(x) G_E^b(y, x) dx dy \\
&= \int_E \mathbb{E}_y \left[\int_0^\infty P_t^{b,E} f(X_s^{b,E}) ds \right] dy \\
&= \int_E \int_t^\infty P_s^{b,E} f(y) ds dy \\
&\leq \int_E \int_0^\infty P_s^{b,E} f(y) ds dy \\
&= \int_E \int_E f(x) G_E^b(y, x) dx dy \\
&= \int_E f(x) \xi_E(dx).
\end{aligned}$$

□

The transition density of the subprocess $X^{b,E}$ with respect to ξ_E is defined by

$$\bar{p}_E^b(t, x, y) := \frac{p_E^b(t, x, y)}{h_E(y)}, \quad \forall (t, x, y) \in (0, \infty) \times E \times E.$$

Then

$$\bar{G}_E^b(x, y) := \int_0^\infty \bar{p}_E^b(t, x, y) dt = \frac{G_E^b(x, y)}{h_E(y)}, \quad \forall x, y \in E$$

is the Green function of $X^{b,E}$ with respect to ξ_E . It is easy to see that $\bar{G}_E^b(x, y)$ has the following properties:

- (A1) $\bar{G}_E^b(x, y) > 0$ on $E \times E$, and $\bar{G}_E^b(x, y) = \infty$ if and only if $x = y$;
- (A2) For every $x \in E$, $\bar{G}_E^b(x, \cdot)$ and $\bar{G}_E^b(\cdot, x)$ are extended continuous in E ;
- (A3) For every compact set $K \subset E$, $\int_K \bar{G}_E^b(x, y) \xi_E(dy) < \infty$.

(A1)-(A3) imply that the process $X^{b,E}$ satisfies the conditions (R) of [13] and conditions (a)(b) of [13, Theorem 5.4]. Thus it satisfies Hunt's Hypothesis (B) by [13, Theorem 5.4]. It follows from [13, Theorem 13.24] that $X^{b,E}$ has a dual process $\hat{X}^{b,E}$ with respect to the reference measure ξ_E , and $\hat{X}^{b,E}$ is a standard process. $\bar{G}_E^b(x, y)$ also satisfies the following properties (A4) and (A5).

- (A4) For every $y \in E$, $\bar{G}_E^b(\cdot, y)$ is an excessive function with respect to $X^{b,E}$, that is, for every $t > 0$ and $x \in E$,

$$\mathbb{E}_x(\bar{G}_E^b(X_t^{b,E}, y)) \leq \bar{G}_E^b(x, y), \text{ and } \lim_{t \downarrow 0} \mathbb{E}_x(\bar{G}_E^b(X_t^{b,E}, y)) = \bar{G}_E^b(x, y).$$

For every $y \in E$, $\bar{G}_E^b(\cdot, y)$ is harmonic with respect to $X^{b,E}$ in $E \setminus \{y\}$. Furthermore, for every open set $U \subset E$, we have

$$\mathbb{E}_x[\bar{G}_E^b(X_{T_U^b}^{b,E}, y)] = \bar{G}_E^b(x, y) \quad \text{for } (x, y) \in E \times U,$$

where $T_U^b := \inf\{t > 0 : X_t^{b,E} \in U\}$. In particular, for every $y \in E$ and $\epsilon > 0$, $\bar{G}_E^b(\cdot, y)$ is regularly harmonic on $E \setminus B(y, \epsilon)$ with respect to $X^{b,E}$.

(A5) For any compact set $K \subset E$ and $y \in E$, $\int_K \bar{G}_E^b(x, y) \xi_E(dx) < \infty$.

Proof of (A4). Using some standard arguments (for example, [10, Proof of (A4)] and the reference therein), we only need to show that for every $x \in E \setminus U$, $\mathbb{E}_x(\bar{G}_E^b(X_{T_U^b}^{b,E}, \cdot))$ is continuous in U . Fix $x \in E \setminus U$ and $y \in U$. Let $r := \delta_U(y)$. For any $\hat{y} \in B(y, r/4)$ and $\delta \in (0, r/2)$, by Lévy system representation of $X^{b,E}$ and (4.11), we have

$$\begin{aligned} & \mathbb{E}_x \left(\bar{G}_E^b(X_{T_U^b}^{b,E}, \hat{y}) : X_{T_U^b}^{b,E} \in B(y, \delta) \right) \\ &= \int_{B(y, \delta)} \bar{G}_E^b(z, \hat{y}) \left(\int_{E \setminus U} G_{E \setminus U}^b(x, w) j^b(w, z) dw \right) dz \\ &= \int_{B(y, \delta)} \frac{G_E^b(z, \hat{y})}{h_E(\hat{y})} \left(\int_{E \setminus U} G_{E \setminus U}^b(x, w) j^b(w, z) dw \right) dz \\ &\leq \frac{c_1}{\inf_{\tilde{y} \in \overline{B(y, r/4)}} h_E(\tilde{y})} \int_{B(y, \delta)} |z - \hat{y}|^{-d+\alpha} \left[\int_{E \setminus U} |x - w|^{-d+\alpha} (|z - w|^{-d-\alpha} + |z - w|^{-d-\beta}) dw \right] dz \\ &\leq \frac{c_2}{\inf_{\tilde{y} \in \overline{B(y, r/4)}} h_E(\tilde{y})} (r^{-d-\alpha} + r^{-d-\beta}) \int_{B(y, \delta)} |z - \hat{y}|^{-d+\alpha} \left[\int_{E \setminus U} |x - w|^{-d+\alpha} dw \right] dz \end{aligned}$$

for some $c_i = c_i(d, \alpha, \beta, E, A) > 0$, $i = 1, 2$. Thus for any $\epsilon > 0$, there exists $\delta \in (0, r/2)$ sufficiently small such that

$$\sup_{\hat{y} \in B(y, r/4)} \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, \hat{y}) : X_{T_U^b}^{b,E} \in B(y, \delta) \right] < \epsilon/8.$$

Fix a sequence $\{y_n\} \subset B(y, r/4)$ such that $y_n \rightarrow y$ as $n \rightarrow \infty$. Since $\bar{G}_E^b(u, v) = G_E^b(u, v)/h_E(v)$ is bounded and jointly continuous in $(E \setminus B(y, \delta)) \times B(y, \delta/2)$, by bounded convergence theorem we have

$$\lim_{n \rightarrow \infty} \left| \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y_n) - \bar{G}_E^b(X_{T_U^b}^{b,E}, y) : X_{T_U^b}^{b,E} \notin B(y, \delta) \right] \right| = 0.$$

Therefore, for n sufficiently large,

$$\begin{aligned} & \left| \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y_n) \right] - \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y) \right] \right| \\ &\leq \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y_n) : X_{T_U^b}^{b,E} \in B(y, \delta) \right] + \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y) : X_{T_U^b}^{b,E} \in B(y, \delta) \right] \\ &\quad + \left| \mathbb{E}_x \left[\bar{G}_E^b(X_{T_U^b}^{b,E}, y_n) - \bar{G}_E^b(X_{T_U^b}^{b,E}, y) : X_{T_U^b}^{b,E} \notin B(y, \delta) \right] \right| \\ &< \epsilon/2. \end{aligned}$$

Hence we complete the proof. \square

Theorem 5.2. For every increasing sequence $\{U_n : n \geq 1\}$ of open sets with $\overline{U}_n \subset U_{n+1}$ and $U_n \uparrow E$, $\lim_{n \rightarrow \infty} \mathbb{E}_x(\bar{G}_E^b(X_{\tau_{U_n}^b}^{b,E}, y)) = 0$ for every $x, y \in E$ with $x \neq y$. Moreover, for every $x, y \in E$, $\lim_{t \rightarrow \infty} \mathbb{E}_x(\bar{G}_E^b(X_t^{b,E}, y)) = 0$.

The proof for the above theorem is much the same as [10, Theorem 5.4], so it is omitted here. Using (A1)-(A5) and Theorem 5.2 we get from [15, 16] that the dual process $\widehat{X}^{b,E}$ is a transient Hunt process. Let $\widehat{P}_t^{b,E}$ denote the semigroup of $\widehat{X}^{b,E}$. Then for every $f, g \in L^2(E, \xi_E(dx))$,

$$\int_E f(x) P_t^{b,E} g(x) \xi_E(dx) = \int_E \widehat{P}_t^{b,E} f(x) g(x) \xi_E(dx). \quad (5.1)$$

Define $\bar{H}_t^E := t$ and

$$\bar{N}^E(x, dy) := \frac{j^b(x, y)}{h_E(y)} \xi_E(dy), \quad \forall (x, y) \in E \times E,$$

$$\bar{N}^E(x, \partial) := \int_{E^c} j^b(x, y) dy, \quad \forall x \in E.$$

Then (\bar{N}^E, \bar{H}^E) is a Lévy system for $X^{b,E}$ with respect to ξ_E . Let $(\widehat{N}^E, \widehat{H}^E)$ denote the Lévy system for $\widehat{X}^{b,E}$ with respect to ξ_E , then it satisfies $\widehat{H}_t^E = t$ and $\widehat{N}^E(y, dx) \xi_E(dy) = \bar{N}^E(x, dy) \xi_E(dx)$. Therefore,

$$\widehat{N}^E(x, dy) = \frac{j^b(y, x)}{h_E(x)} \xi_E(dy) = \frac{j^b(y, x) h_E(y)}{h_E(x)} dy, \quad \forall (x, y) \in E \times E,$$

$$\widehat{N}^E(x, \partial) = \int_{E^c} \frac{j^b(y, x) h_E(y)}{h_E(x)} dy, \quad \forall x \in E.$$

For any open subset U of E , let $\widehat{X}^{b,E,U}$ denote the subprocess of $\widehat{X}^{b,E}$ in U . Then $X^{b,U}$ and $\widehat{X}^{b,E,U}$ are dual processes with respect to $\xi_E(dx)$. By the duality relation (5.1), we have the following theorem.

Theorem 5.3. *For any open subset U in E ,*

$$\widehat{p}_U^{b,E}(t, x, y) := \frac{p_U^b(t, y, x) h_E(y)}{h_E(x)}$$

is jointly continuous on $[0, \infty) \times U \times U$, and it is the transition density of $\widehat{X}^{b,E,U}$ with respect to Lebesgue measure. Moreover,

$$\widehat{G}_U^{b,E}(x, y) := \int_0^\infty \widehat{p}_U^{b,E}(t, x, y) dt = \frac{G_U^b(y, x) h_E(y)}{h_E(x)}, \quad \forall (x, y) \in U \times U$$

is the Green function of $\widehat{X}^{b,E,U}$ with respect to the Lebesgue measure.

6 Small time heat kernel estimates

In this section we assume that D is a bounded $C^{1,1}$ open set in \mathbb{R}^d and that E is a ball centered at the origin such that $D \subset \frac{1}{4}E$. We also assume that b is a bounded function satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A < \infty$. Define

$$M = M(A, E) := \sup \left\{ \sup_{x, y \in \frac{3}{4}E} \frac{h_E(x)}{h_E(y)} : b \text{ satisfies (1.2) and (1.4) with } \|b\|_\infty \leq A \right\}. \quad (6.1)$$

M is a scale-invariant constant in the sense that $M_\lambda := M(A_\lambda, \lambda E) = M(A, E)$ for every $\lambda > 0$. Clearly $M \geq 1$. The finiteness of M follows from Lemma 4.2, domain monotonicity of Green functions, and Theorem 4.11 if the radius of E is large. We observe that by taking the radius of E to be $4\text{diam}(D)$, the constant M depends on d, α, β, A and D with dependence on D via the diameter of D .

6.1 Small time upper bound estimates

For an open subset U of E , let $\hat{\tau}_U^{b,E} := \inf\{t > 0 : \hat{X}_t^{b,E} \notin U\}$. The proof of the following lemma is much the same as [10, Lemma 7.3], we omit the details here.

Lemma 6.1. *Suppose U is a open subset of $\frac{1}{4}E$. U_1, U_3 are open subsets of U with $\text{dist}(U_1, U_3) > 0$ and $U_2 = U \setminus (U_1 \cup U_3)$. Then for any $x \in U_1, y \in U_3$ and $t > 0$, we have*

$$p_U^b(t, x, y) \leq \mathbb{P}_x \left(X_{\tau_{U_1}^b}^b \in U_2 \right) \sup_{\substack{s < t \\ z \in U_2}} p_U^b(s, z, y) + \left(t \wedge \mathbb{E}_x(\tau_{U_1}^b) \right) \text{esssup}_{\substack{u \in U_1 \\ z \in U_3}} j^b(u, z). \quad (6.2)$$

$$p_U^b(t, x, y) \leq M \mathbb{P}_x \left(\hat{X}_{\hat{\tau}_{U_1}^{b,E}}^{b,E} \in U_2 \right) \sup_{\substack{s < t \\ z \in U_2}} p_U^b(s, y, z) + M \left(t \wedge \mathbb{E}_x(\hat{\tau}_{U_1}^{b,E}) \right) \text{esssup}_{\substack{u \in U_3 \\ z \in U_1}} j^b(u, z). \quad (6.3)$$

$$p_U^b(1/3, x, y) \geq \frac{1}{3M} \mathbb{P}_x \left(\tau_{U_1}^b > \frac{1}{3} \right) \mathbb{P}_y \left(\hat{\tau}_{U_3}^{b,E} > \frac{1}{3} \right) \text{essinf}_{\substack{u \in U_1 \\ z \in U_3}} j^b(u, z). \quad (6.4)$$

Lemma 6.2. *Let U be an arbitrary $C^{1,1}$ open subset of $\frac{1}{4}E$ with $\text{diam}(U) \leq r_3$ where r_3 is the constant in Lemma 4.13. Then*

$$\mathbb{P}_x \left(\hat{X}_{\hat{\tau}_U^{b,E}}^{b,E} \in \partial U \right) = 0, \quad \forall x \in U.$$

Proof. Fix $x \in U$. Let $r = \frac{1}{2}(\delta_U(x) \wedge r_1)$. Through similar arguments as in the beginning of the proof for Lemma 4.13, we can find a ball $B := B(x, r/2) \subset U$ and a ball $B' \subset E \cap \{z \in \bar{U}^c : \text{dist}(z, U) < \varepsilon(A)\}$ with radius and distance to B comparable with r . Since $|z - y| < \varepsilon(A)$ for every $z \in B$ and $y \in B'$, it follows from Theorem 4.11, Theorem 5.3 and (3.4) that

$$\begin{aligned} \mathbb{P}_x \left(\hat{X}_{\hat{\tau}_B^{b,E}}^{b,E} \in B' \right) &= \int_{B'} \int_B \hat{G}_B^{b,E}(x, y) j^b(z, y) \frac{h_E(z)}{h_E(y)} dy dz \\ &= \int_{B'} \int_B G_B^b(y, x) j^b(z, y) \frac{h_E(z)}{h_E(x)} dy dz \\ &\geq M^{-1} \int_{B'} \int_B G_B^b(y, x) j^b(z, y) dy dz \\ &\geq \frac{1}{4} M^{-1} \int_{B'} \int_B G_B(x, y) j(y, z) dy dz \\ &= \frac{1}{4} M^{-1} \mathbb{P}_x(X_{\tau_B} \in B') \geq c > 0 \end{aligned}$$

for some constant $c = c(d, \alpha) > 0$. Thus we can apply similar arguments as in Lemma 4.13 to get the conclusion. \square

Lemma 6.3. *There exists a scale-invariant positive constant $C_{23} = C_{23}(d, \alpha, \beta, D, A, M)$ such that for all $x \in D$ with $\delta_D(x) < (r_1 \wedge r_2 \wedge r_3)/16$, we have*

$$\mathbb{P}_x \left(\tau_D^b > \frac{1}{4} \right) \leq C_{23}(1 \wedge \delta_D(x)^{\alpha/2}), \quad (6.5)$$

$$\mathbb{P}_x \left(\hat{\tau}_D^{b,E} > \frac{1}{4} \right) \leq C_{23}(1 \wedge \delta_D(x)^{\alpha/2}). \quad (6.6)$$

Proof. We only give the proof of (6.6). The proof of (6.5) is similar. Let $r_* = r_1 \wedge r_2 \wedge r_3$. Let $Q_x \in \partial D$ be such that $|x - Q_x| = \delta_D(x)$. Denote $U = V(Q_x, r_*/4)$ such that $D \cap B(Q_x, r_*/8) \subset U \subset D \cap B(Q_x, r_*/2)$. Then by Lemma 4.12, Theorem 5.3 and Lemma 6.2 we have

$$\begin{aligned} & \mathbb{P}_x \left(\hat{\tau}_D^{b,E} > \frac{1}{4} \right) \\ & \leq \mathbb{P}_x \left(\hat{\tau}_U^{b,E} > \frac{1}{4} \right) + \mathbb{P}_x \left(\hat{X}_{\hat{\tau}_U^{b,E}}^{b,E} \in D \right) \\ & \leq 4\mathbb{E}_x(\hat{\tau}_U^{b,E}) + \mathbb{P}_x \left(\hat{X}_{\hat{\tau}_U^{b,E}}^{b,E} \in D \right) \\ & = 4 \int_U \hat{G}_U^{b,E}(x, y) dy + \int_{D \setminus U} \int_U \hat{G}_U^{b,E}(x, y) \frac{j^b(z, y) h_E(z)}{h_E(y)} dy dz \\ & = 4 \int_U G_U^b(y, x) \frac{h_E(y)}{h_E(x)} dy + \int_{D \setminus U} \int_U G_U^b(y, x) j^b(z, y) \frac{h_E(z)}{h_E(x)} dy dz \\ & \leq 6M \int_U G_U(y, x) dy + \frac{3}{2}M \int_{D \setminus U} \int_U G_U(y, x) j^b(z, y) dy dz \\ & \leq 6M \int_U G_U(x, y) dy + \frac{3}{2}M(1 + \text{Adiam}(D)^{\alpha-\beta}) \int_{D \setminus U} \int_U G_U(y, x) j(z, y) dy dz \\ & = 6M\mathbb{E}_x(\tau_U) + \frac{3}{2}M(1 + \text{Adiam}(D)^{\alpha-\beta})\mathbb{P}_x(X_{\tau_U} \in D \setminus U) \\ & \leq c_1 \delta_U(x)^{\alpha/2} = c_1 \delta_D(x)^{\alpha/2} \end{aligned} \quad (6.7)$$

for some scale invariant constant $c_1 = c_1(d, \alpha, \beta, D, A, M) > 0$. The assertion follows immediately from (6.7) and the fact that $\mathbb{P}_x \left(\hat{\tau}_D^{b,E} > 1/4 \right) \leq 1$. \square

Lemma 6.4. *There exists a positive constant $C_{24} = C_{24}(d, \alpha, \beta, D, A, M)$ such that for any $x, y \in D$,*

$$p_D^b(1/2, x, y) \leq C_{24}(1 \wedge \delta_D(x)^{\alpha/2}) \left(1 \wedge \frac{1}{|x - y|^{d+\alpha}} \right), \quad (6.8)$$

$$p_D^b(1/2, x, y) \leq C_{24}(1 \wedge \delta_D(y)^{\alpha/2}) \left(1 \wedge \frac{1}{|x - y|^{d+\alpha}} \right). \quad (6.9)$$

Moreover C_{24} satisfies that

$$C_{24}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \leq (1 \vee \lambda^{-d-\frac{3}{2}\alpha}) C_{24}(d, \alpha, \beta, D, A, M)$$

for every $\lambda > 0$.

Proof. We only need to prove (6.9). The proof of (6.8) is similar. Let $r_* = 1 \wedge r_1 \wedge r_2 \wedge r_3$. By (2.1) and the domain monotonicity, we get

$$p_D^b(t, x, y) \leq p^b(t, x, y) \leq c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \quad \forall x, y \in D, \quad t \in (0, 1]$$

for some constant $c_1 = c_1(d, \alpha, \beta, D, A) > 0$. This together with (4.12) and the scaling property for p_D^b imply that

$$p_D^b(t, x, y) \leq c_2 \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \quad \forall x, y \in D, \quad t \in (0, \infty) \quad (6.10)$$

for some scale-invariant constant $c_2 = c_2(d, \alpha, \beta, D, A) > 0$. Immediately if $\delta_D(y) \geq r_*/16$, then

$$\begin{aligned} p_D^b(1/2, x, y) &\lesssim c_2 \left(1 \wedge |x - y|^{-d-\alpha} \right) \\ &= c_2 \left(1 \vee \delta_D(y)^{-\alpha/2} \right) \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right) \\ &\lesssim c_2 \left(1 \vee r_*^{-\alpha/2} \right) \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right) \\ &= c_2 r_*^{-\alpha/2} \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right) \end{aligned} \quad (6.11)$$

Now we consider $\delta_D(y) < r_*/16$. For every $x, y \in D$ with $|x - y|/8 < r_*$, by Theorem 5.3, (6.10) and Lemma 6.3 we have

$$\begin{aligned} p_D^b(1/2, x, y) &= \int_D p_D^b(1/4, x, z) p_D^b(1/4, z, y) dz \\ &= \int_D p_D^b(1/4, x, z) \hat{p}_D^{b,E}(1/4, y, z) \frac{h_E(y)}{h_E(z)} dz \\ &\lesssim M \int_D \left(1 \wedge |x - z|^{-d-\alpha} \right) \hat{p}_D^{b,E}(1/4, y, z) dz \\ &\leq M \mathbb{P}_x \left(\hat{\tau}_D^{b,E} > 1/4 \right) \\ &\leq C_{23} M^2 \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \\ &= C_{23} M^2 \left(1 \vee |x - y|^{d+\alpha} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right) \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \\ &\lesssim C_{23} M^2 \left(1 \vee r_*^{d+\alpha} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right) \left(1 \wedge \delta_D(y)^{\alpha/2} \right) \\ &= C_{23} M^2 \left(1 \wedge |x - y|^{-d-\alpha} \right) \left(1 \wedge \delta_D(y)^{\alpha/2} \right). \end{aligned} \quad (6.12)$$

Next we consider $x, y \in D$ with $|x - y|/8 \geq r_*$. Let $Q_y \in \partial D$ be such that $|y - Q_y| = \delta_D(y)$. Let $U_y := V(Q_y, r_*/2)$ be a $C^{1,1}$ domain such that $D \cap B(Q_y, r_*/4) \subset U_y \subset D \cap B(Q_y, r_*)$. Denote $D_3 = \{z \in D : |z - y| > |x - y|/2\}$ and $D_2 = D \setminus (U_y \cup D_3)$. Note that by (6.3) we have

$$\begin{aligned} p_D^b(1/2, x, y) &\leq M \mathbb{P}_y \left(\hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in D_2 \right) \sup_{s < 1/2, z \in D_2} p_D^b(s, x, z) \\ &\quad + M \left(\frac{1}{2} \wedge \mathbb{E}_y(\hat{\tau}_{U_y}^{b,E}) \right) \text{esssup}_{u \in U_y, z \in D_3} j^b(z, u). \end{aligned} \quad (6.13)$$

For every $z \in D_3$ and $u \in U_y$, we have $|u - z| \geq |z - y| - |u - y| \geq |x - y|/2 - 2r_* \geq |x - y|/4$, and consequently,

$$\begin{aligned}
& \text{esssup}_{u \in U_y, z \in D_3} j^b(z, u) \\
&= \text{esssup}_{u \in U_y, z \in D_3} \left(\frac{\mathcal{A}(d, -\alpha)}{|u - z|^{d+\alpha}} + \mathcal{A}(d, -\beta) \frac{b(z, u - z)}{|u - z|^{d+\beta}} \right) \\
&\lesssim (1 + \text{Adiam}(D)^{\alpha-\beta}) |x - y|^{-d-\alpha} \\
&= (1 + \text{Adiam}(D)^{\alpha-\beta}) (1 \vee |x - y|^{-d-\alpha}) (1 \wedge |x - y|^{-d-\alpha}) \\
&\lesssim (1 + \text{Adiam}(D)^{\alpha-\beta}) r_*^{-d-\alpha} (1 \wedge |x - y|^{-d-\alpha}). \tag{6.14}
\end{aligned}$$

For any $z \in D_2$, we have $|z - x| \geq |x - y| - |y - z| \geq |x - y|/2 > 4r_*$, thus

$$\begin{aligned}
\sup_{s < 1/2, z \in D_2} p_D^b(s, x, z) &\lesssim \sup_{s < 1/2, z \in D_2} \left(s^{-d/\alpha} \wedge \frac{s}{|x - z|^{d+\alpha}} \right) \\
&\lesssim |x - y|^{-d-\alpha} \\
&\lesssim r_*^{-d-\alpha} (1 \wedge |x - y|^{-d-\alpha}). \tag{6.15}
\end{aligned}$$

By (6.13), (6.14) and (6.15), we have

$$p_D^b(1/2, x, y) \leq c_4 M r_*^{-d-\alpha} (1 \wedge |x - y|^{-d-\alpha}) \left[\mathbb{P}_y \left(\widehat{X}_{\widehat{\tau}_{U_y}^{b,E}}^{b,E} \in D_2 \right) + \mathbb{E}_y(\widehat{\tau}_{U_y}^{b,E}) \right]$$

for some scale-invariant constant $c_4 = c_4(d, \alpha, \beta, D, A) > 0$. By Lemma 4.12, we have

$$\begin{aligned}
& \mathbb{P}_y \left(\widehat{X}_{\widehat{\tau}_{U_y}^{b,E}}^{b,E} \in D_2 \right) + \mathbb{E}_y(\widehat{\tau}_{U_y}^{b,E}) \\
&= \int_{D_2} \int_{U_y} G_{U_y}^b(w, y) j^b(z, w) \frac{h_E(z)}{h_E(y)} dw dz + \int_{U_y} G_{U_y}^b(w, y) \frac{h_E(w)}{h_E(y)} dw \\
&\leq c_5 (1 + \text{Adiam}(D)^{\alpha-\beta}) M \int_{D_2} \int_{U_y} G_{U_y}(w, y) j(z, w) dw dz + \frac{3}{2} M \int_{U_y} G_{U_y}(y, w) dw \\
&\leq c_6 r_*^{-\alpha/2} \delta_{U_y}(y)^{\alpha/2} = c_6 r_*^{-\alpha/2} \delta_D(y)^{\alpha/2}
\end{aligned}$$

for some scale-invariant positive constants $c_5 = c_5(d, \alpha, \beta)$ and $c_6 = c_6(d, \alpha, \beta, D, A, M)$. Therefore for every $x, y \in D$ with $|x - y|/8 \geq r_*$, there is a scale-invariant constant $c_7 = c_7(d, \alpha, \beta, D, A, M) > 0$ such that

$$p_D^b(1/2, x, y) \leq c_7 r_*^{-d-\frac{3}{2}\alpha} \delta_D(y)^{\alpha/2} (1 \wedge |x - y|^{-d-\alpha}). \tag{6.16}$$

Combine (6.11), (6.12) and (6.16), we have

$$p_D^b(1/2, x, y) \leq c_8 (1 \vee r_*^{-d-\frac{3}{2}\alpha}) (1 \wedge \delta_D(y)^{\alpha/2}) (1 \wedge |x - y|^{-d-\alpha})$$

for some scale-invariant constant $c_8 = c_8(d, \alpha, \beta, D, A, M) > 0$. Hence we complete the proof by setting $C_{24} = c_8 (1 \vee r_*^{-d-\frac{3}{2}\alpha})$. In this case C_{24} satisfies that $C_{24}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \leq (1 \vee \lambda^{-d-\frac{3}{2}\alpha}) C_{24}(d, \alpha, \beta, D, A, M)$ for any $\lambda > 0$. \square

Lemma 6.5. *There exists a constant $C_{25} = C_{25}(d, \alpha, \beta, D, A, M) > 0$ such that*

$$p_D^b(1, x, y) \leq C_{25}(1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2})(1 \wedge |x - y|^{-d-\alpha}), \quad \forall x, y \in D.$$

Moreover C_{25} satisfies that

$$C_{25}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \leq (1 \vee \lambda^{-2d-3\alpha})C_{25}(d, \alpha, \beta, D, A, M)$$

for any $\lambda > 0$.

Proof. By semigroup property and Lemma 6.4, we have

$$\begin{aligned} & p_D^b(1, x, y) \\ &= \int_D p_D^b(1/2, x, z) p_D^b(1/2, z, y) dz \\ &\leq C_{24}^2 (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \int_{\mathbb{R}^d} (1 \wedge |x - z|^{-d-\alpha})(1 \wedge |z - y|^{-d-\alpha}) dz \\ &\leq c_1 C_{24}^2 (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \int_{\mathbb{R}^d} p(1/2, x, z) p(1/2, z, y) dz \\ &= c_1 C_{24}^2 (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) p(1, x, y) \\ &\leq c_1 c_2 C_{24}^2 (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2})(1 \wedge |x - y|^{-d-\alpha}) \end{aligned}$$

for some positive constants $c_i = c_i(d, \alpha, \beta)$, $i = 1, 2$. Hence we complete the proof by setting $C_{25} = c_1 c_2 C_{24}^2$. \square

Theorem 6.6. *For every $0 < T < \infty$, there is a positive constant $C_{26} = C_{26}(d, \alpha, \beta, D, A, M, T)$ such that for every $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D^b(t, x, y) \leq C_{26} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).$$

Proof. Set $\lambda = t^{-1/\alpha}$, by the scaling property (4.3) and Lemma 6.5, we get

$$\begin{aligned} & p_D^b(t, x, y) \\ &= \lambda^{-d} p_{\lambda D}^b(1, \lambda x, \lambda y) \\ &\leq C_{25}(d, \alpha, \beta, \lambda D, A_\lambda, M_\lambda) \lambda^{-d} (1 \wedge \delta_{\lambda D}(\lambda x)^{\alpha/2})(1 \wedge \delta_{\lambda D}(\lambda y)^{\alpha/2})(1 \wedge |\lambda x - \lambda y|^{-d-\alpha}) \\ &\leq (1 \vee t^{3+2d/\alpha}) C_{25}(d, \alpha, \beta, D, A, M) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \\ &\leq (1 \vee T^{3+2d/\alpha}) C_{25}(d, \alpha, \beta, D, A, M) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \end{aligned}$$

Hence we complete the proof. \square

6.2 Small time lower bound estimates

The next proposition follows directly from Proposition 3.4 and Theorem 5.3.

Proposition 6.7. *For any $a_1 \in (0, 1)$, $a_3 > a_2 > 0$, $A > 0$ and $R \in (0, 1/2]$, there exists a positive constant $C_{27} = C_{27}(d, \alpha, \beta, a_1, a_2, a_3, R, A)$ such that for every $x_0 \in \mathbb{R}^d$ and $B(x_0, r) \subset \frac{3}{4}E$ with $0 < r \leq R$, we have*

$$\hat{p}_{B(x_0, r)}^{b, E}(t, x, y) \geq C_{27} M^{-1} r^{-d} \quad \text{for } x, y \in B(x_0, a_1 r), \quad t \in [a_2 r^\alpha, a_3 r^\alpha]. \quad (6.17)$$

Moreover, if b satisfies (1.6) for some constant $\varepsilon > 0$, then (6.17) holds for all $R > 0$ and some constant $C_{27} = C_{27}(d, \alpha, \beta, a_1, a_2, a_3, R, A, \varepsilon) > 0$.

Corollary 6.8. *For any $a_1 \in (0, 1)$ and $r \in (0, 1/2]$, there exists a positive constant $C_{28} = C_{28}(d, \alpha, \beta, a_1, r, A)$ such that*

$$\begin{aligned} p_{B(x_0, r)}^b(1/3, x, y) &\geq C_{28} r^{-d}, \quad \forall x, y \in B(x_0, a_1 r), \\ \hat{p}_{B(x_0, r)}^{b, E}(1/3, x, y) &\geq C_{28} M^{-1} r^{-d}, \quad \forall x, y \in B(x_0, a_1 r). \end{aligned}$$

Moreover, if b satisfies (1.6) for some constant $\varepsilon > 0$, then the above estimates hold for all $r > 0$ and some $C_{28} = C_{28}(d, \alpha, \beta, a_1, r, A, \varepsilon) > 0$.

Lemma 6.9. *Suppose D is a bounded $C^{1,1}$ open set. There is a positive constant $C_{29} = C_{29}(d, \alpha, \beta, D, A, M)$ that is scale-invariant in D in the sense that $C_{29}(d, \alpha, \beta, \lambda D, A, M) = C_{29}(d, \alpha, \beta, D, A, M)$ for any $\lambda \geq 1$ so that for every $x, y \in D$ with $|x - y| < \frac{4}{5}\varepsilon(A)$,*

$$p_D^b(1, x, y) \geq C_{29} \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \left(1 \wedge |x - y|^{-d-\alpha}\right).$$

Moreover, if b satisfies (1.6) for some constant $\varepsilon > 0$, then the above estimate holds for all $x, y \in D$ and some $C_{29} = C_{29}(d, \alpha, \beta, D, A, M, \varepsilon) > 0$ that is scale-invariant in D .

Proof. Suppose D is a $C^{1,1}$ open set with characteristic (R_0, Λ_0) and scale r_0 . There exist scale-invariant constants $\delta_0 = \delta_0(R_0, \Lambda_0) \in (0, r_0/8)$ and $L_0 = L_0(R_0, \Lambda_0) > 1$ such that for all $x, y \in D$, there are $\xi_x \in D \cap B(x, L_0 \delta_0)$ and $\xi_y \in D \cap B(y, L_0 \delta_0)$ with $B(\xi_x, 2\delta_0) \cap B(x, 2\delta_0) = \emptyset$, $B(\xi_y, 2\delta_0) \cap B(y, 2\delta_0) = \emptyset$ and $B(\xi_x, 8\delta_0) \cup B(\xi_y, 8\delta_0) \subset D$. Set $\delta = \delta(D, A) := (1 \wedge \delta_0 \wedge r_1 \wedge r_2 \wedge \frac{\varepsilon(A)}{2L_0+8})/10$. Obviously, δ is scale-invariant in D . By the semigroup property, we have

$$\begin{aligned} p_D^b(1, x, y) &\geq \int_{v \in B(\xi_y, \delta)} \int_{u \in B(\xi_x, \delta)} p_D^b(1/3, x, u) p_D^b(1/3, u, v) p_D^b(1/3, v, y) du dv \\ &\geq \left(\int_{u \in B(\xi_x, \delta)} p_D^b(1/3, x, u) du \right) \left(\int_{v \in B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \right) \\ &\quad \left(\operatorname{essinf}_{\substack{u \in B(\xi_x, \delta) \\ v \in B(\xi_y, \delta)}} p_D^b(1/3, u, v) \right). \end{aligned} \quad (6.18)$$

First we claim that there is a positive constant $c_1 = c_1(d, \alpha, \beta, D, A, M)$ which is scale-invariant in D , such that for every $x, y \in D$ with $|x - y| < \frac{4}{5}\varepsilon(A)$,

$$\operatorname{essinf}_{\substack{u \in B(\xi_x, \delta) \\ v \in B(\xi_y, \delta)}} p_D^b(1/3, u, v) \geq c_1(1 \wedge |x - y|^{-d-\alpha}). \quad (6.19)$$

Moreover, if b also satisfies (1.6) for some constant $\varepsilon > 0$, then (6.19) holds for every $x, y \in D$ and some $c_1 = c_1(d, \alpha, \beta, D, A, M, \varepsilon) > 0$ that is scale-invariant in D .

Fix $x, y \in D$, $u \in B(\xi_x, \delta)$ and $v \in B(\xi_y, \delta)$. Since $\delta_D(\xi_x), \delta_D(\xi_y) > 8\delta$, then $\delta_D(u), \delta_D(v) > 7\delta$. If $|u - v| \leq 2\delta < 1/5$, then by the domain monotonicity and Corollary 6.8, we have

$$p_D^b(1/3, u, v) \geq p_{B(u, 3\delta)}^b(1/3, u, v) \geq c_2 \geq c_2(1 \wedge |x - y|^{-d-\alpha}) \quad (6.20)$$

for some $c_2 = c_2(d, \alpha, \beta, A) > 0$. If $|u - v| > 2\delta$, then $\operatorname{dist}(B(u, \delta), B(v, \delta)) > 0$. By (6.4) and Corollary 6.8,

$$\begin{aligned} & p_D^b(1/3, u, v) \\ & \geq \frac{1}{3M} P_u(\tau_{B(u, \delta)}^b > 1/3) \mathbb{P}_v(\widehat{\tau}_{B(v, \delta)}^{b, E} > 1/3) \left(\operatorname{essinf}_{\substack{w \in B(u, \delta) \\ z \in B(v, \delta)}} j^b(w, z) \right) \\ & \geq \frac{1}{3M} \left(\int_{B(u, \delta/2)} p_{B(u, \delta)}^b(1/3, u, y) dy \right) \left(\int_{B(v, \delta/2)} \widehat{p}_{B(v, \delta)}^{b, E}(1/3, v, y) dy \right) \left(\operatorname{essinf}_{\substack{w \in B(u, \delta) \\ z \in B(v, \delta)}} j^b(w, z) \right) \\ & \geq c_3 M^{-2} \operatorname{essinf}_{\substack{w \in B(u, \delta) \\ z \in B(v, \delta)}} j^b(w, z) \end{aligned}$$

for some constant $c_3 = c_3(d, \alpha, \beta, A) > 0$. Since for every $x, y \in D$ with $|x - y| < \frac{4}{5}\varepsilon(A)$, $w \in B(u, \delta)$ and $z \in B(v, \delta)$,

$$|w - z| \leq |\xi_x - \xi_y| + 4\delta \leq |x - y| + 2L_0\delta + 4\delta < \varepsilon(A)$$

and $|w - z| \leq |u - v| + 2\delta \leq 2|u - v|$. Thus we have by (3.4)

$$\begin{aligned} p_D^b(1/3, u, v) & \geq c_4 M^{-2} \operatorname{essinf}_{\substack{w \in B(u, \delta) \\ z \in B(v, \delta)}} |w - z|^{-d-\alpha} \\ & \geq c_5 M^{-2} |u - v|^{-d-\alpha} \geq c_5 M^{-2} (1 \wedge |u - v|^{-d-\alpha}), \end{aligned} \quad (6.21)$$

where $c_i = c_i(d, \alpha, \beta, A) > 0$, $i = 4, 5$. If $x, y \in D$ and $|x - y| \geq \delta/8$, then $|u - v| \leq |x - y| + (2L_0 + 2)\delta \leq (16L_0 + 17)|x - y|$ for every $u \in B(\xi_x, \delta)$ and $v \in B(\xi_y, \delta)$, and consequently

$$1 \wedge |u - v|^{-d-\alpha} \geq c_6(1 \wedge |x - y|^{-d-\alpha}) \quad (6.22)$$

for some constant $c_6 = c_6(L_0) > 0$. If $|x - y| < \delta/8$, then $|u - v| \leq (2L_0 + 17/8)\delta$ for every $u \in B(\xi_x, \delta)$ and $v \in B(\xi_y, \delta)$. Note that $\delta < 1$, immediately we get

$$1 \wedge |u - v|^{-d-\alpha} \geq c_7 \geq c_7(1 \wedge |x - y|^{-d-\alpha}) \quad (6.23)$$

for some constant $c_7 = c_7(L_0) > 0$. Therefore, (6.19) follows from (6.20), (6.21), (6.22) and (6.23). When b also satisfies (1.6), (6.21) is then true for every $x, y \in D$, every $u \in B(\xi_x, \delta)$,

$v \in B(\xi_y, \delta)$ and some $c_5 = c_5(d, \alpha, \beta, A, \varepsilon) > 0$. The above argument shows that (6.19) holds for all $x, y \in D$. This proves the claim.

Next we claim that there is a positive constant $c_8 = c_8(d, \alpha, \beta, D, A, M)$ which is scale-invariant in D , such that for every $x, y \in D$

$$\int_{B(\xi_x, \delta)} p_D^b(1/3, x, u) du \geq c_8(1 \wedge \delta_D(x)^{\alpha/2}), \quad (6.24)$$

$$\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \geq c_8(1 \wedge \delta_D(y)^{\alpha/2}). \quad (6.25)$$

We only give a proof for (6.25). The proof of (6.24) is similar. First we consider $y \in D$ with $\delta_D(y) < \delta$. Let $Q \in \partial D$ be such that $|y - Q| = \delta_D(y)$. Let U_y be the $C^{1,1}$ domain in D with characteristic $(2\delta R_0/L, \Lambda_0 L/2\delta)$ such that $D \cap B(Q, 2\delta) \subset U_y \subset D \cap B(Q, 4\delta)$. Denote $V_y := D \cap B(Q, 6\delta)$. Since $\text{dist}(B(\xi_y, \delta), V_y) > 0$, we have by (6.4)

$$\begin{aligned} \int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv &\geq \frac{1}{3M} \left(\int_{B(\xi_y, \delta)} \mathbb{P}_v(\tau_{B(\xi_y, \delta)}^b > 1/3) dv \right) \mathbb{P}_y(\hat{\tau}_{V_y}^{b,E} > 1/3) \\ &\quad \left(\text{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in V_y}} j^b(w, z) \right). \end{aligned} \quad (6.26)$$

Since $\delta < 1/10$, it follows from Corollary 6.8 that

$$\begin{aligned} \int_{B(\xi_y, \delta)} \mathbb{P}_v(\tau_{B(\xi_y, \delta)}^b > 1/3) dv &= \int_{B(\xi_y, \delta)} \int_{B(\xi_y, \delta)} p_{B(\xi_y, \delta)}^b(1/3, v, w) dw dv \\ &\geq \int_{B(\xi_y, \delta/2)} \int_{B(\xi_y, \delta/2)} p_{B(\xi_y, \delta)}^b(1/3, v, w) dw dv \\ &\geq c_9 \delta^d \end{aligned} \quad (6.27)$$

where $c_9 = c_9(d, \alpha, \beta, A) > 0$. Note that $\delta \leq |z - w| \leq (L_0 + 8)\delta < \varepsilon(A)$ for every $w \in B(\xi_y, \delta)$ and $z \in V_y$. Thus by (3.4),

$$\begin{aligned} \text{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in V_y}} j^b(w, z) &\geq \frac{1}{2} \text{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in V_y}} \bar{j}_{\varepsilon(A)}(w, z) \\ &\geq \frac{1}{2} \frac{\mathcal{A}(d, -\alpha)}{((L_0 + 8)\delta)^{d+\alpha}} \geq c_{10} \delta^{-d} \end{aligned} \quad (6.28)$$

where $c_{10} = c_{10}(d, \alpha, L_0) > 0$. Since D is bounded and $C^{1,1}$, there is a ball $B(y_0, 2c_{11}\delta)$ in

$D \cap (B(Q, 6\delta) \setminus B(Q, 4\delta))$ for some constant $c_{11} = c_{11}(d, \Lambda_0) \in (0, 1)$. Thus

$$\begin{aligned}
& \mathbb{P}_y \left(\hat{\tau}_{V_y}^{b,E} > 1/3 \right) \\
& \geq \mathbb{P}_y \left(\hat{\tau}_{V_y}^{b,E} > 1/3, \hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in B(y_0, c_{11}\delta/2) \right) \\
& = \mathbb{E}_y \left(P_{\hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E}} \left(\hat{\tau}_{V_y}^{b,E} > 1/3 \right); \hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in B(y_0, c_{11}\delta/2) \right) \\
& \geq \mathbb{E}_y \left[P_{\hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E}} \left(\hat{\tau}_{B(\hat{X}_{\hat{\tau}_{U_y}^{b,E}, c_{11}\delta})}^{b,E} > 1/3 \right); \hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in B(y_0, c_{11}\delta/2) \right] \\
& \geq \inf_{w \in B(y_0, c_{11}\delta/2)} \mathbb{P}_w \left(\hat{\tau}_{B(w, c_{11}\delta)}^{b,E} > 1/3 \right) \mathbb{P}_y \left(\hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in B(y_0, c_{11}\delta/2) \right). \tag{6.29}
\end{aligned}$$

It follows from Corollary 6.8 that for every $w \in B(y_0, c_{11}\delta/2)$

$$\begin{aligned}
\mathbb{P}_w \left(\hat{\tau}_{B(w, c_{11}\delta)}^{b,E} > 1/3 \right) & \geq \int_{B(w, c_{11}\delta/2)} \hat{p}_{B(w, c_{11}\delta)}^{b,E}(1/3, w, y) dy \\
& \geq c_{12}M^{-1} \tag{6.30}
\end{aligned}$$

where $c_{12} = c_{12}(d, \alpha, \beta, A, L_0) > 0$. Note that $|w - z| < 10\delta \leq \varepsilon(A)$ for every $w \in U_y \subset B(Q, 4\delta)$ and $z \in B(y_0, c_{11}\delta/2) \subset B(Q, 6\delta)$. Thus by Lemma 4.12 and (3.4),

$$\begin{aligned}
\mathbb{P}_y \left(\hat{X}_{\hat{\tau}_{U_y}^{b,E}}^{b,E} \in B(y_0, c_{11}\delta/2) \right) & = \int_{B(y_0, c_{11}\delta/2)} \int_{U_y} G_{U_y}^b(w, y) j^b(z, w) \frac{h_E(z)}{h_E(y)} dw dz \\
& \geq \frac{1}{4M} \int_{B(y_0, c_{11}\delta/2)} \int_{U_y} G_{U_y}(y, w) j(w, z) dw dz \\
& = \frac{1}{4M} \mathbb{P}_y \left(X_{\tau_{U_y}} \in B(y_0, c_{11}\delta/2) \right) \\
& \geq c_{13}\delta^{-\alpha/2} \delta_{U_y}(y)^{\alpha/2} = c_{13}\delta^{-\alpha/2} \delta_D(y)^{\alpha/2} \\
& \geq c_{13}\delta_D(y)^{\alpha/2} \tag{6.31}
\end{aligned}$$

for some constant $c_{13} = c_{13}(d, \alpha, \beta, \Lambda_0, R_0) > 0$. By (6.26), (6.27), (6.28), (6.29), (6.30) and (6.31) we conclude that for every $y \in D$ with $\delta_D(y) < \delta$,

$$\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \geq c_{14}\delta_D(y)^{\alpha/2} \geq c_{14}(1 \wedge \delta_D(y)^{\alpha/2}) \tag{6.32}$$

for some positive constant $c_{14} = \frac{1}{3M^2} c_9 c_{10} c_{12} c_{13}$ which is scale-invariant in D . On the other hand if $y \in D$ with $\delta_D(y) \geq \delta$, then since $\text{dist}(B(\xi_y, \delta), B(y, \delta)) > 0$, by (6.4) we have

$$\begin{aligned}
\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv & \geq \frac{1}{3M} \left(\int_{B(\xi_y, \delta)} \mathbb{P}_v \left(\tau_{B(\xi_y, \delta)}^b > 1/3 \right) dv \right) \mathbb{P}_y \left(\hat{\tau}_{B(y, \delta)}^{b,E} > 1/3 \right) \\
& \quad \left(\text{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in B(y, \delta)}} j^b(w, z) \right). \tag{6.33}
\end{aligned}$$

Similarly as in (6.27) and (6.29), we have

$$\int_{B(\xi_y, \delta)} \mathbb{P}_v \left(\tau_{B(\xi_y, \delta)}^b > 1/3 \right) dv \geq \int_{B(\xi_y, \delta/2)} \int_{B(\xi_y, \delta/2)} p_{B(\xi_y, \delta)}^b(1/3, v, w) dw dv \geq c_{15} \delta^d, \quad (6.34)$$

and

$$\begin{aligned} \mathbb{P}_y \left(\hat{\tau}_{B(y, \delta)}^{b, E} > 1/3 \right) &\geq \int_{B(y, \delta/2)} \hat{p}_{B(y, \delta)}^{b, E}(1/3, y, w) dw \\ &\geq c_{16} M^{-1} \end{aligned} \quad (6.35)$$

for some positive constants $c_i = c_i(d, \alpha, \beta, A)$, $i = 15, 16$. Note that for every $w \in B(\xi_y, \delta)$ and $z \in B(y, \delta)$, $|w - z| \leq (L_0 + 2)\delta < \varepsilon(A)$. Thus

$$\begin{aligned} \operatorname{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in B(y, \delta)}} j^b(w, z) &\geq \frac{1}{2} \operatorname{essinf}_{\substack{w \in B(\xi_y, \delta) \\ z \in B(y, \delta)}} \bar{j}_\varepsilon(w, z) \\ &\geq \frac{1}{2} \frac{\mathcal{A}(d, \alpha)}{((L_0 + 2)\delta)^{d+\alpha}} \geq c_{17} \delta^{-d} \end{aligned} \quad (6.36)$$

where $c_{17} = c_{17}(d, \alpha, L_0) > 0$. By (6.33)-(6.36),

$$\int_{B(\xi_y, \delta)} p_D^b(1/3, v, y) dv \geq c_{18} \geq c_{18}(1 \wedge \delta_D(y)^{\alpha/2}) \quad \text{for } y \in D \text{ with } \delta_D(y) \geq \delta$$

with $c_{18} = c_{15}c_{16}c_{17}/(3M^2)$ which is scale-invariant in D . This together with (6.32) establishes (6.25). \square

Theorem 6.10. *Suppose D is a bounded $C^{1,1}$ open set and $T \in (0, \infty)$. There exists a positive constant $C_{30} = C_{30}(d, \alpha, \beta, D, A, M, T)$ that is scale-invariant in D so that for every $x, y \in D$ with $|x - y| < \frac{4}{5}\varepsilon(A)$ and $t \in (0, T]$,*

$$p_D^b(t, x, y) \geq C_{30} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).$$

Moreover, if b also satisfies (1.6) for some $\varepsilon > 0$, then the above estimate holds for all $x, y \in D$, all $t \in (0, T]$ and some positive constant $C_{30} = C_{30}(d, \alpha, \beta, D, A, M, T, \varepsilon)$.

Proof. For any $t \in (0, T]$, set $\lambda = t^{-1/\alpha}$. Recall that $b_\lambda(x, z) := \lambda^{\beta-\alpha} b(x/\lambda, y/\lambda)$. Clearly $\|b_\lambda\|_\infty = \lambda^{\beta-\alpha} \|b\|_\infty \leq T^{1-\beta/\alpha} A$. Since $|\lambda x - \lambda y| < 4\lambda\varepsilon(A)/5 \leq 4\varepsilon(T^{1-\beta/\alpha}A)/5$ for every $x, y \in D$ with $|x - y| < 4\varepsilon(A)/5$, it follows from the scaling property for p_D^b and Lemma 6.9 that for $|x - y| < 4\varepsilon(A)/5$,

$$\begin{aligned} &p_D^b(t, x, y) \\ &= \lambda^{-d} p_{\lambda D}^{b_\lambda}(1, \lambda x, \lambda y) \\ &\geq C_{29}(d, \alpha, \beta, \lambda D, T^{1-\beta/\alpha} A, M_\lambda) (1 \wedge \delta_{\lambda D}(\lambda x)^{\alpha/2}) (1 \wedge \delta_{\lambda D}(\lambda y)^{\alpha/2}) (1 \wedge |\lambda x - \lambda y|^{-d-\alpha}) \\ &= C_{29}(d, \alpha, \beta, D, T^{1-\beta/\alpha} A, M) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \\ &\geq C_{29}(d, \alpha, \beta, D, T^{1-\beta/\alpha} A, M) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \end{aligned}$$

When b also satisfies condition (1.6) for some $\varepsilon > 0$, the above estimate holds for any $x, y \in D$ as so does the estimate in Lemma 6.9. \square

Corollary 6.11. *Suppose D is a bounded $C^{1,1}$ open set with $\text{diam}(D) \leq \frac{4}{5}\varepsilon(A)$ and $T \in (0, \infty)$. There exists a positive constant $C_{31} = C_{31}(d, \alpha, \beta, D, A, M, T)$ such that for every $x, y \in D$ and $t \in (0, T]$,*

$$p_D^b(t, x, y) \geq C_{31} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).$$

Theorem 6.12. *Suppose D is a connected bounded $C^{1,1}$ open set and $T \in (0, \infty)$. There exists a positive constant $C_{32} = C_{32}(d, \alpha, \beta, D, A, M, T)$ such that for every $x, y \in D$ and $t \in (0, T]$,*

$$p_D^b(t, x, y) \geq C_{32} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).$$

Proof. Suppose (R_0, Λ_0) is the $C^{1,1}$ characteristic of D . Let $t_0 := \frac{4}{5}\varepsilon(A)$. Fix $x, y \in D$. In the rest of this proof, we use $d(x, y)$ to denote the path distance between x and y in D . First we claim that for any $a_2 > a_1 > 0$, there is a positive constant $c_1 = c_1(d, \alpha, \beta, a_1, a_2, D, A, M)$ which is scale-invariant in D , such that for all $t \in [a_1 t_0^\alpha, a_2 t_0^\alpha]$ and $x, y \in D$,

$$p_D^b(t, x, y) \geq c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (6.37)$$

It follows from Theorem 6.10 that the above lower bound is true for $x, y \in D$ with $d(x, y) < t_0$ or $|x - y| < t_0$. Now we consider $x, y \in D$ with $t_0 \leq d(x, y) < 3t_0/2$ and $|x - y| \geq t_0$. Let z be the midpoint of the path in D connected x and y . Immediately $|z - x| \vee |z - y| \leq 3t_0/4$. Let $r := \frac{1}{8}t_0 \wedge R_0$. By Proposition 2.2 there exists a ball $B_0 := B(A, \theta r) \subset D \cap B(z, r)$ for some constant $\theta = \theta(\Lambda_0) \in (0, 1)$. Let $B_1 := B(A, \theta r/2)$. Fix $w_1, w_2 \in B(A, \theta r/4)$ and $w_1 \neq w_2$. Note that for every $w \in B_0$, $|x - w| \leq |w - w_1| + t_0$ and $|y - w| \leq |w - w_2| + t_0$. For every $t \in [a_1 t_0^\alpha, a_2 t_0^\alpha]$, we have $t_0 + (t/2)^{1/\alpha} \stackrel{c_2(a_1, a_2)}{\asymp} (t/2)^{1/\alpha}$, and thus

$$\begin{aligned} \left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|x - w|^{d+\alpha}} &\geq \left(\frac{t}{2} \right)^{-d/\alpha} \left(1 \wedge \frac{(t/2)^{1/\alpha}}{|w - w_1| + t_0} \right)^{d+\alpha} \\ &\asymp \left(\frac{t}{2} \right)^{-d/\alpha} \left(\frac{(t/2)^{1/\alpha}}{|w - w_1| + t_0 + (t/2)^{1/\alpha}} \right)^{d+\alpha} \\ &\stackrel{c_2(a_1, a_2)}{\asymp} \left(\frac{t}{2} \right)^{-d/\alpha} \left(\frac{(t/2)^{1/\alpha}}{|w - w_1| + (t/2)^{1/\alpha}} \right)^{d+\alpha} \\ &\asymp \left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_1|^{d+\alpha}}. \end{aligned} \quad (6.38)$$

Similarly we can prove that

$$\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|y - w|^{d+\alpha}} \stackrel{c_3(a_1, a_2)}{\gtrsim} \left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_2|^{d+\alpha}}. \quad (6.39)$$

Note that $|x - w| \vee |y - w| < t_0$ for every $w \in B_1$. Thus for every $t \in [a_1 t_0^\alpha, a_2 t_0^\alpha]$, by Theorem 6.10, (6.38) and (6.39) we have

$$\begin{aligned}
& p_D^b(t, x, y) \\
& \geq \int_{B_1} p_D^b(t/2, x, w) p_D^b(t/2, w, y) dw \\
& \geq c_4 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/2}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/2}} \right) \\
& \quad \int_{B_1} \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/2}} \right)^2 \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|x - w|^{d+\alpha}} \right) \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - y|^{d+\alpha}} \right) dw \\
& \geq c_5 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \\
& \quad \int_{B_1} \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_1|^{d+\alpha}} \right) \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_2|^{d+\alpha}} \right) dw \\
& \geq c_6 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \int_{B_1} p_{B_1}(t/2, w_1, w) p_{B_1}(t/2, w, w_2) dw \\
& = c_6 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) p_{B_1}(t, w_1, w_2) \\
& \geq c_7 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_{B_1}(w_1)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_{B_1}(w_2)^{\alpha/2}}{\sqrt{t}} \right) \\
& \quad \left(t^{-d/\alpha} \wedge \frac{t}{|w_1 - w_2|^{d+\alpha}} \right) \\
& \geq c_8 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \tag{6.40}
\end{aligned}$$

where $c_i = c_i(d, \alpha, \beta, a_1, a_2, D, A, M) > 0$, $i = 4, \dots, 8$, and the last inequality is because $\delta_{B_1}(w_1), \delta_{B_1}(w_2) \geq \theta r/4$ and $|w_1 - w_2| \leq \theta r/2 \leq t_0/8 \leq |x - y|/8$. Inductively by semigroup property we can prove that (6.37) holds for every $t \in [a_1 t_0^\alpha, a_2 t_0^\alpha]$, every $n \in \mathbb{N}$ and $x, y \in D$ with $d(x, y) < nt_0/2$. Since D is bounded and connected $C^{1,1}$ open set, there is scale-invariant constants $c_9 = c_9(D) \geq 1$ and $k = k(D) \in \mathbb{N}$ such that for every $x, y \in D$, $d(x, y) \leq c_9|x - y| \leq c_9 \text{diam}(D) \leq kt_0/2$. Therefore the assertion can be generalized to every $t \in [a_1 t_0^\alpha, a_2 t_0^\alpha]$ and every $x, y \in D$ by repeating the above arguments.

For any $t \in (0, T]$, set $\lambda = t_0 t^{-1/\alpha}$. Then by the scaling property and (6.37), we have

$$\begin{aligned}
& p_D^b(t, x, y) \\
&= \lambda^d p_{\lambda D}^{b_\lambda}(t_0^\alpha, \lambda x, \lambda y) \\
&\geq c_1(d, \alpha, \beta, 1, 2, \lambda D, t_0^{\beta-\alpha} T^{1-\beta/\alpha} A, M_\lambda) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\
&\geq c_1(d, \alpha, \beta, 1, 2, D, t_0^{\beta-\alpha} T^{1-\beta/\alpha} A, M) \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).
\end{aligned}$$

The theorem is proved. \square

Theorem 6.13. Suppose D is a bounded $C^{1,1}$ open set and $T \in (0, \infty)$. D_1 and D_2 are two connected components of D with $\text{dist}(D_1, D_2) < \frac{4}{5}\varepsilon(A)$. Then there exists a positive constant $C_{33} = C_{33}(d, \alpha, \beta, D, A, M, T)$ such that for every $t \in (0, T]$, $x \in D_1$ and $y \in D_2$, we have

$$p_D^b(t, x, y) \geq C_{33} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Proof. Let $x_0 \in \partial D_1$ and $y_0 \in \partial D_2$ be such that $|x_0 - y_0| = \text{dist}(D_1, D_2)$. Set $r := \frac{1}{4}(\frac{4}{5}\varepsilon(A) - |x_0 - y_0|) \wedge R_0$. Choose ball $B_1 := B(A_1, \kappa r) \subset D_1 \cap B(x_0, r)$ and $B_2 := B(A_2, \kappa r) \subset D_2 \cap B(y_0, r)$ for some constant $\kappa = \kappa(\Lambda_0) \in (0, 1)$.

Case I: If $x \in D_1 \cap B(x_0, r)$ and $y \in D_2 \cap B(y_0, r)$, then $|x - y| < 4\varepsilon/5$. The assertion is immediately true by Theorem 6.10.

Case II: If $x \in D_1 \setminus B(x_0, r)$ and $y \in D_2 \cap B(y_0, r)$, without loss of generality we may assume $|x - y| \geq 4\varepsilon/5$. For all $a_2 > a_1 > 0$, every $w_1, w_2 \in B(A_1, \kappa r/4)$ with $w_1 \neq w_2$, every $w \in B_1$, and $t \in [a_1 \text{diam}(D)^\alpha, a_2 \text{diam}(D)^\alpha]$, we have

$$\begin{aligned}
\left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|x-w|^{d+\alpha}} &\geq \left(\frac{t}{2}\right)^{-d/\alpha} \left(1 \wedge \frac{(t/2)^{1/\alpha}}{|w_1-w| + \text{diam}(D)}\right)^{d+\alpha} \\
&\asymp \left(\frac{t}{2}\right)^{-d/\alpha} \left(\frac{(t/2)^{1/\alpha}}{|w_1-w| + (t/2)^{1/\alpha} + \text{diam}(D)}\right)^{d+\alpha} \\
&\stackrel{c_1(a_1, a_2)}{\asymp} \left(\frac{t}{2}\right)^{-d/\alpha} \left(1 \wedge \frac{(t/2)^{1/\alpha}}{|w_1-w| + (t/2)^{1/\alpha}}\right)^{d+\alpha} \\
&\asymp \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w_1-w|^{d+\alpha}}, \tag{6.41}
\end{aligned}$$

and similarly

$$\left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|y-w|^{d+\alpha}} \stackrel{c_2(a_1, a_2)}{\gtrsim} \left(\frac{t}{2}\right)^{-d/\alpha} \wedge \frac{t/2}{|w_2-w|^{d+\alpha}}. \tag{6.42}$$

Let $B_3 := B(A_1, \kappa r/2)$. Note that for every $w \in B_3$, $|y - w| < \frac{4}{5}\varepsilon(A)$. By Theorem 6.12, Theorem 6.10, (6.38) and (6.39), we have for every $t \in [a_1 \text{diam}(D)^\alpha, a_2 \text{diam}(D)^\alpha]$

$$\begin{aligned}
& p_D^b(t, x, y) \\
& \geq \int_{B_3} p_{D_1}^b(t/2, x, w) p_D^b(t/2, w, y) dw \\
& \geq c_3 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/2}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/2}} \right) \\
& \quad \int_{B_3} \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/2}} \right)^2 \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|x - w|^{d+\alpha}} \right) \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - y|^{d+\alpha}} \right) dw \\
& \geq c_4 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \\
& \quad \int_{B_3} \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_1|^{d+\alpha}} \right) \left(\left(\frac{t}{2} \right)^{-d/\alpha} \wedge \frac{t/2}{|w - w_2|^{d+\alpha}} \right) dw \\
& \geq c_5 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \int_{B_3} p_{B_3}(t/2, w_1, w) p_{B_3}(t/2, w, w_2) dw \\
& = c_5 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) p_{B_3}(t, w_1, w_2) \\
& \geq c_6 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_{B_3}(w_1)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_{B_3}(w_2)^{\alpha/2}}{\sqrt{t}} \right) \\
& \quad \left(t^{-d/\alpha} \wedge \frac{t}{|w_1 - w_2|^{d+\alpha}} \right) \\
& \geq c_7 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \tag{6.43}
\end{aligned}$$

where $c_i = c_i(d, \alpha, \beta, a_1, a_2, D, A, M) > 0$, $i = 3, \dots, 7$. Using the scaling property, we can generalize the assertion to all $t \in (0, T]$.

Case III: If $x \in D_1 \setminus B(x_0, r)$ and $y \in D_2 \setminus B(y_0, r)$, note that

$$p_D^b(t, x, y) \geq \int_{B(A_2, \kappa r/2)} p_D^b(t/2, x, w) p_{D_2}^b(t/2, w, y) dw.$$

We can apply similar arguments as in Case II here and prove the assertion. \square

7 Large time heat kernel estimates

We recall the facts from spectral theory. Let \mathcal{A} be a linear operator defined on a linear subspace $D(\mathcal{A})$ of a Banach space Y . Its resolvent set $\rho(\mathcal{A})$ is the collection of all complex number $\lambda \in \mathbb{C}$ so that $(\lambda I - \mathcal{A})^{-1}$ exists as a bounded linear operator on Y . It is known that $\rho(\mathcal{A})$ is an open set in \mathbb{C} . The spectrum set $\sigma(\mathcal{A})$ is defined to be $\mathbb{C} \setminus \rho(\mathcal{A})$.

We assume that E is a open ball in \mathbb{R}^d centered at the origin and $D \subset \frac{1}{4}E$ an arbitrary open set. Define

$$P_t^{b,D} f(x) := \int_D f(y) p_D^b(t, x, y) dx, \quad f \in L^2(D; dx).$$

Since for every $t > 0$, $(x, y) \mapsto p_D^b(t, x, y)$ is bounded on $D \times D$, it follows that

$$\int_{D \times D} p_D^b(t, x, y)^2 dx dy = \int_D p_D^b(2t, x, x) dx < \infty.$$

So for each $t > 0$, $P_t^{b,D}$ is a Hilbert-Schmidt operator, and hence compact. Thus by Riesz-Schauder theorem, $\sigma(P_t^{b,D})$ is a discrete set that has limit point 0, and each non-zero $\lambda \in \sigma(P_t^{b,D})$ is an eigenvalue of finite multiplicity. We use (\cdot, \cdot) and $\|\cdot\|_2$ to denote the inner product and norm in $L^2(D; dx)$, respectively.

Theorem 7.1. *There exist positive constants $\lambda_0 = \lambda_0(d, \alpha, \beta, D, A)$ and $C_{34} = C_{34}(d, \alpha, \beta, D, A)$ so that*

$$\mathbb{P}_x(\tau_D^b > t) \leq C_{34} e^{-\lambda_0 t} \quad \text{for every } x \in D \text{ and } t > 0. \quad (7.1)$$

Furthermore, $\lambda_1^{b,D} := -\sup \operatorname{Re} \sigma(\mathcal{L}^{b,D}) \geq \lambda_0$ and there is a positive continuous function ϕ on D with unit $L^2(D; dx)$ -norm so that

$$P_t^{b,D} \phi = e^{-t\lambda_1^{b,D}} \phi \quad \text{for every } t > 0. \quad (7.2)$$

Moreover, $\sigma(\mathcal{L}^{b,D})$ is a discrete set consisting of eigenvalues that has no limit points, and $-\lambda_1^{b,D}$ is an eigenvalue of $\mathcal{L}^{b,D}$ with $-\lambda_1^{b,D} > \operatorname{Re} \mu$ for any other $\mu \in \sigma(\mathcal{L}^{b,D})$.

Proof. Since for each $t > 0$, $P_t^{b,D}$ is a compact operator, by [18, Proposition V.6.6], its spectral radius $r_t := \sup \{|\lambda| : \lambda \in \sigma(P_t^{b,D})\} > 0$ is an eigenvalue of $P_t^{b,D}$ with a unique eigenfunction $\phi^{(t)}$ with unit L^2 -norm and $\phi^{(t)} > 0$ a.e. on D . Moreover, if λ is another eigenvalue of $P_t^{b,D}$, then $|\lambda| < r_t$. Observe that for $z \in \mathbb{C}$ and integer $k \geq 1$, $z - P_{kt}^{b,D} = \prod_{j=1}^k (z_j - P_t^{b,D})$ where $\{z_j; 1 \leq j \leq k\}$ are the complex k -th roots of z . It follows that for any $t > 0$ and $k \geq 1$,

$$r_{kt} = r_t^k \quad \text{and} \quad \phi^{(kt)} = \phi^{(t)},$$

the latter follows from the semigroup property $P_{kt}^{b,D} \phi^{(t)} = r_t^k \phi^{(t)}$ and the uniqueness of eigenfunction corresponding to λ_{kt} . (The above conclusion can also be deduced from (7.4) below.) Let $\phi := \phi^{(1)}$ and $\lambda_1 := -\log r_1$. Then we conclude from the above display that $r_t = r_1^t = e^{-\lambda_1 t}$ and $\phi^{(t)} = \phi$ for every rational number $t > 0$. Consequently, $P_t^{b,D} \phi = e^{-\lambda_1 t} \phi$ for every rational $t > 0$ and hence for every $t > 0$ in view of Theorem 3.2. The latter theorem together with Proposition 2.1 implies that $\phi = e^{\lambda_1} P_1^{b,D} \phi$ is a bounded positive continuous function on D . Clearly we have for $t > 0$ and $x \in D$,

$$|\phi(x)| \leq \|\phi\|_\infty e^{\lambda_1 t} P_t^{b,D} 1(x) = \|\phi\|_\infty e^{\lambda_1 t} \mathbb{P}_x(\tau_D^b > t). \quad (7.3)$$

By Proposition 2.1, $\inf_{x \in D} \mathbb{P}_x(\tau_D^b \leq 1) \geq \inf_{x \in D} \int_{D^c} p^b(1, x, y) dy \geq \varepsilon_0 > 0$, where ε_0 depends only on d, α, β and A . Consequently, $\sup_{x \in D} \mathbb{P}_x(\tau_D^b > 1) \leq 1 - \varepsilon_0$. It follows from the Markov

property of X^b that $\sup_{x \in D} \mathbb{P}_x(\tau_D^b > n) \leq (1 - \varepsilon_0)^n$. This establishes (7.1) with $\lambda_0 := -\log(1 - \varepsilon_0)$ and $C_{34} = e^{\lambda_0}$. Moreover, it follows from (7.3) that $\lambda_1 \geq \lambda_0$.

Recall that $\mathcal{L}^{b,D}$ denotes the infinitesimal generator of $P_t^{b,D}$ in $L^2(D; dx)$. From above, clearly ϕ is an eigenfunction of $\mathcal{L}^{b,D}$ with eigenvalue $-\lambda_0$. Since each $P_t^{b,D}$ is compact, each resolvent operator $(\lambda I - \mathcal{L}^{b,D})^{-1}$ with $\lambda \in \rho(\mathcal{L}^{b,D})$ is compact (cf. [17, Theorem II.3.3]). Fix some $\lambda \in \rho(\mathcal{L}^{b,D})$. By Riesz-Schauder theorem, $\sigma((\lambda - \mathcal{L}^{b,D})^{-1})$ is a discrete set that has limit point 0, and each non-zero point in $\sigma((\lambda - \mathcal{L}^{b,D})^{-1})$ is an eigenvalue of finite multiplicity. It follows that $\sigma(\mathcal{L}^{b,D})$ is a discrete set consisting of eigenvalues that converges to $+\infty$ and each eigenvalue is of finite multiplicity. We also know by [17, Theorem 2.4] that

$$e^{t\sigma(\mathcal{L}^{b,D})} \subset \sigma(P_t^{b,D}) \subset e^{t\sigma(\mathcal{L}^{b,D})} \cup \{0\}. \quad (7.4)$$

It follows then $\lambda_1 = -\sup \operatorname{Re} \sigma(\mathcal{L}^{b,D})$. \square

The large time heat kernel estimate for $p_D^b(t, x, y)$ can be obtained in a similar way as that in [6].

7.1 Large time upper bound estimate

Theorem 7.2. *Suppose D is an arbitrary bounded $C^{1,1}$ open set in \mathbb{R}^d and $A, T \in (0, \infty)$. Let $\lambda_0 > 0$ and $\lambda_1^{b,D} \geq \lambda_0$ be as in Theorem 7.1. Then there are positive constants $C_{35} = C_{35}(d, \alpha, \beta, D, A, T) > 0$ and $C_{36} = C_{36}(d, \alpha, \beta, D, A, b, T) > 0$ so that for every bounded function b satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A$, we have*

$$p_D^b(t, x, y) \leq C_{35} e^{-t\lambda_0} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}, \quad (t, x, y) \in [T, \infty) \times D \times D. \quad (7.5)$$

and

$$p_D^b(t, x, y) \leq C_{36} e^{-t\lambda_1^{b,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}, \quad (t, x, y) \in [T, \infty) \times D \times D. \quad (7.6)$$

Proof. Without loss of generality, we assume $T = 1$. Let ϕ be the positive eigenfunction in Theorem 7.1, and $r_1 = 4\varepsilon(A)/5$ the constant in Theorem 6.10. First, for $t > 1$, we have by the Chapman-Kolmogorov equation, Theorem 6.6 and Theorem 7.1,

$$\begin{aligned} p^{b,D}(t, x, y) &= \int_{D \times D} p^{b,D}(1/2, x, z) p^{b,D}(t-1, z, w) p^{b,D}(1/2, w, y) dz dw \\ &\leq c_1 (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_{D \times D} p^{b,D}(t-1, z, w) dz dw \\ &\leq c_1 C_{34} (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} e^{-\lambda_0(t-1)} |D|, \end{aligned}$$

where $c_1 = c_1(d, \alpha, \beta, D, A) > 0$. This proves (7.5).

By the geometric property of the $C^{1,1}$ open set D , there is a constant $\kappa \in (0, 1)$ so that for every $x \in D$, there is a point $A(x)$ so that $B(A(x), \kappa r_1) \subset B(x, r_1) \cap D$. We know from Theorem

7.1 that $\lambda_1^{b,D} > 0$. For notational simplicity, we write λ_1 for $\lambda_1^{b,D}$ in this proof.

$$\begin{aligned}
\phi(x) &= e^{\lambda_1} P_1^{b,D} \phi(x) \\
&\geq c_2 e^{\lambda_1} (1 \wedge \delta_D(x))^{\alpha/2} \int_{B(x, r_1) \cap D} (1 \wedge \delta_D(y))^{\alpha/2} \left(1 \wedge \frac{1}{|x-y|^{d+\alpha}} \right) \phi(y) dy \\
&\geq c_2 e^{\lambda_1} (1 \wedge \delta_D(x))^{\alpha/2} \left(1 \wedge r_1^{-(d+\alpha)} \right) \int_{B(A(x), \kappa r_1)} (1 \wedge \delta_D(y))^{\alpha/2} \phi(y) dy \\
&\geq c_3 e^{\lambda_1} (1 \wedge \delta_D(x))^{\alpha/2}.
\end{aligned} \tag{7.7}$$

Here $c_2 = c_2(d, \alpha, \beta, D, A) > 0$ and $c_3 = c_3(d, \alpha, \beta, D, A, b) > 0$. The last inequality is due to the fact that $v(z) := \int_{B(z, \kappa r_1/2)} (1 \wedge \delta_D(y))^{\alpha/2} \phi(y) dy$ is a positive continuous function on the compact set $\{z \in D : \delta_D(z) \geq \kappa r_1/2\}$ and its minimum there is strictly positive. For $t > 1$, by the Chapman-Kolmogorov equation, Theorem 6.6, Theorem 7.1 and (7.7),

$$\begin{aligned}
p^{b,D}(t, x, y) &= \int_{D \times D} p^{b,D}(1/2, x, z) p^{b,D}(t-1, z, w) p^{b,D}(1/2, w, y) dz dw \\
&\leq c_4 (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_{D \times D} p^{b,D}(t-1, z, w) (1 \wedge \delta_D(w))^{\alpha/2} dz dw \\
&\leq c_4 c_3^{-1} e^{-\lambda_1} (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_{D \times D} p^{b,D}(t-1, z, w) \phi(w) dz dw \\
&= c_4 c_3^{-1} e^{-\lambda_1} (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_D e^{-\lambda_1(t-1)} \phi(z) dz \\
&\leq c_5 c_3^{-1} e^{-\lambda_1 t} (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2}.
\end{aligned}$$

Here $c_i = c_i(d, \alpha, \beta, D, A) > 0$, $i = 4, 5$. This establishes (7.6). \square

7.2 Large time lower bound estimate

Theorem 7.3. *Suppose D is a bounded $C^{1,1}$ open set and b is a bounded function satisfying (1.2) and (1.4) with $\|b\|_\infty \leq A < \infty$. Assume also that D and b satisfy one of the following assumptions:*

- (i) $\text{diam}(D) < 4\varepsilon(A)/5$;
- (ii) D is connected;
- (iii) $\text{dist}(D_i, D_j) < 4\varepsilon(A)/5$ for every connected components D_i, D_j of D ;
- (iv) b satisfies (1.6) for some $\varepsilon > 0$.

Then for every $T \in (0, \infty)$, there exists a constant $C_{37} = C_{37}(d, \alpha, \beta, D, A, M, T, \varepsilon) \geq 1$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,

$$C_{37}^{-1} e^{-t\lambda_1^{b,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \leq p_D^b(t, x, y) \leq C_{37} e^{-t\lambda_1^{b,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}.$$

Here $\lambda_1^{b,D} := -\sup \text{Re } \sigma(\mathcal{L}^{b,D}) > 0$.

Proof. For notational simplicity, we write λ_1 for $\lambda_1^{b,D}$ in this proof. Let ϕ be the positive eigenfunction in Theorem 7.1. By Theorem 6.6 and Hölder inequality we have for every $x \in D$,

$$\begin{aligned}
\phi(x) &= e^{\lambda_1} P_1^{b,D} \phi(x) \\
&= e^{\lambda_1} \int_D \phi(y) p_D^b(1, x, y) dy \\
&\leq c_1 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \int_D \phi(y) dy \\
&\leq c_1 e^{\lambda_1} |D|^{1/2} \|\phi\|_2 (1 \wedge \delta_D(x)^{\alpha/2}) \\
&=: c_2 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2})
\end{aligned} \tag{7.8}$$

where $c_i = c_i(d, \alpha, \beta, D, A, M) > 0$, $i = 1, 2$. By the lower bound estimates for p_D^b established in Section 6.2 and (7.8), under our assumptions we have for every $x \in D$

$$\begin{aligned}
\phi(x) &= e^{3\lambda_1} P_3^{b,D} \phi(x) \\
&= e^{3\lambda_1} \int_D \phi(y) p_D^b(3, x, y) dy \\
&\geq c_3 e^{3\lambda_1} (1 \wedge \text{diam}(D)^{-d-\alpha}) \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \int_D \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \phi(y) dy \\
&=: c_4 e^{2\lambda_1} \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \int_D e^{\lambda_1} \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \phi(y) dy \\
&\geq c_4 c_2^{-1} e^{2\lambda_1} \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \int_D \phi(y)^2 dy \\
&= c_4 c_2^{-1} e^{2\lambda_1} \left(1 \wedge \delta_D(x)^{\alpha/2}\right).
\end{aligned} \tag{7.9}$$

where $c_i = c_i(d, \alpha, \beta, D, A, M) > 0$, $i = 3, 4$. Recall that $\lambda_1 > 0$. By (7.8) and (7.9), we get

$$1 \leq e^{\lambda_1} \leq c_2^2 c_4^{-1} := c_5. \tag{7.10}$$

Applying similar calculations as in (7.9) to $\phi(x) = e^{\lambda_1} P_1^{b,D} \phi(x)$, we get

$$\phi(x) \geq c_6 e^{\lambda_1} (1 \wedge \delta_D(x)^{\alpha/2}) \quad \text{for } c_6 = c_6(d, \alpha, \beta, D, A, M) > 0. \tag{7.11}$$

Note that for every $t > 0$,

$$\begin{aligned}
1 = \int_D \phi(x)^2 dx &= e^{\lambda_1 t} \int_D \phi(x) P_t^{b,D} \phi(x) dx \\
&= e^{\lambda_1 t} \int_D \int_D \phi(x) p_D^b(t, x, y) \phi(y) dx dy.
\end{aligned}$$

This together with (7.8), (7.10) and (7.11) implies that

$$c_2^{-2} c_5^{-2} e^{-\lambda_1 t} \leq \int_D \int_D (1 \wedge \delta_D(x)^{\alpha/2}) p_D^b(t, x, y) (1 \wedge \delta_D(y)^{\alpha/2}) dx dy \leq c_6^{-2} e^{-\lambda_1 t}. \tag{7.12}$$

By the Chapman-Kolmogorov equation, two-sided estimates for p_D^b established in Section 6, (7.12) and (7.10), we have

$$\begin{aligned}
& p_D^b(t, x, y) \\
&= \int_D \int_D p_D^b(T/4, x, z) p_D^b(t - T/2, z, w) p_D^b(T/4, w, y) dw dz \\
&\asymp \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{T/4}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{T/4}}\right) \left(\left(\frac{T}{4}\right)^{-d/\alpha} \wedge \frac{T}{(\text{diam} D)^{d+\alpha}}\right)^2 \\
&\quad \int_D \int_D \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{T/4}}\right) p_D^b(t - T/2, z, w) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{T/4}}\right) dw dz \\
&\asymp (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \int_D \int_D (1 \wedge \delta_D(z)^{\alpha/2}) p_D^b(t - T/2, z, w) (1 \wedge \delta_D(w)^{\alpha/2}) dz dw \\
&\asymp e^{-\lambda_1(t - \frac{T}{2})} (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) \\
&\asymp e^{-\lambda_1 t} (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}).
\end{aligned}$$

This completes the proof. \square

The following follows immediately from Theorem 7.3 and the domain monotonicity.

Theorem 7.4. *Let D be a bounded $C^{1,1}$ open subset of \mathbb{R}^d and $T \in (0, \infty)$. There exists a positive constant $C_{38} = C_{38}(d, \alpha, \beta, D, A, M, T)$ such that for every $x, y \in D$ with $|x - y| < 4\varepsilon(A)/5$ and $t \in (T, \infty)$,*

$$p_D^b(t, x, y) \geq C_{38} e^{-\lambda_1^{b, D_x \cup D_y} t} \delta_D(x) \delta_D(y),$$

where D_x denotes the connected component containing x and $\lambda_1^{b, D_x \cup D_y} := -\sup \text{Re } \sigma(\mathcal{L}^{b, D_x \cup D_y}) > 0$.

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Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA.

E-mail: zqchen@uw.edu

Ting Yang

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081,
P.R. China.

Email: yangt@bit.edu.cn